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Localisation and damping in resonators with complex geometry

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Abstract. Based on numerical studies, we show that localisation is a common phenomenon in resonators exhibiting some kind of geometrical complexity. In two-dimensional (2d) shallow cavities of irregular shape, localisation effects are due to spatial decoherence in a major fraction of the volume. In 2d shallow cavities of regular geometry with embedded absorbing material of irregular shape, one observes the appearance of eigenmodes localised in both, the absorbing and the non-absorbing media. Those modes are thought to be responsible for increased dissipation. These results may be a hint to understand why natural or practical systems absorbing wave energy are found, or built, with complex geometry.

1 Introduction

It is a common fact that resonators with a simple and smooth geometry are ‘good’ resonators. A good resonator is a physical system which, for specific frequencies, is able to accumulate a large reactive energy at the expense of only small power dissipation. This is widely used in microwave, radar, antenna and laser technology as well as in musical instruments. On the opposite, when one wants to absorb wave energy, one frequently builds systems with irregular shapes. This is true for anechoic chambers, acoustic or electromagnetic, or for breakwaters optimised to absorb sea-wave energy. Up to now, these facts are used in a purely empirical manner through experimental or numerical optimisation but no theoretical concept has yet arisen that could explain these facts easily. In the following we show, considering several examples, that localisation appears as soon as a resonator exhibits complex geometry. But as shown below, localisation implies larger energy dissipation, so that resonators with complex geometry are more damped than regularly shaped resonators.

2 How localisation is quantified

In the following we discuss the properties of the lower eigenmodes of the Helmholtz equation in cavities with irregular shapes and in simple rectangular cavities containing an absorbing material with an irregular shape. A few of the eigenmode amplitude distributions are shown in the figures throughout this paper. One observes that these eigenstates strongly differ from usual sine functions. They are ‘localised’ or confined in some subregion of the resonator. Beginning with acoustic cavities without damping, the eigenmodes are the solutions of the eigenvalue problem

$$\nabla^2\psi_n = -k_n^2\psi_n, \tag{1}$$

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with the Neumann boundary condition at the walls

$$\mathbf{n} \cdot \boldsymbol{\nabla}\psi_n = 0, \quad \mathbf{n} \text{ the normal vector to the boundary.} \tag{2}$$

The eigenvalue k_n of the above problem is the ratio of the eigenfrequency ω_n divided by the sound velocity.

For 2d shallow cavities Ω with boundary $\partial\Omega$, the localisation or confinement of the eigenmode ψ_n is characterised by its "existence surface" [1,2]:

$$S_n = \frac{1}{\int_\Omega |\psi_n|^4 \, \mathrm{d}S} \tag{3}$$

where ψ_n is normalised by

$$\int_\Omega |\psi_n|^2 \, \mathrm{d}S = 1. \tag{4}$$

According to this definition, which corresponds to the inverse participation ratio in electronic localisation, a mode will be considered as being *localised* if its existence surface S_n is significantly smaller than the surface area $S = a^2$ of the cavity. In a square (or rectangular) cavity, with classical cosine eigenmodes ψ_{mn} ($m, n \in \mathbb{N}$), the relative existence surface S_{mn}/S is respectively equal to 1 for $m = n = 0$, to 2/3 for m or $n = 0$, and to 4/9 for m and $n > 0$. There are no localised modes in such a particular geometry.

3 Localisation in acoustical cavities with complex geometry

A few examples of localised modes in various irregular shallow cavities are shown in Fig. 1. For the six different irregular cavities, one particular localised mode has been selected. For all these cases, one observes the strong confinement of the amplitude distribution in the vicinity of the irregular wall. More details are given in [3]. Localisation was first found in prefractal cavities but fractality is not a prerequisite for localisation: any irregular cavity will exhibit localised modes, whatever the shape and characteristic sizes of its boundary.

The insets in Fig. 1 give for each cavity, the histogram of the relative existence surface S_n/S of the eigenmodes in the frequency range $k_n a/\pi \in [0, 10]$, compared to the results for a square cavity. The effect of the irregularity clearly appears on these plots: in each case, apart from the fundamental mode whose relative existence surface is equal to 1 independently of the geometry, almost all of the modes have a relative existence surface smaller than 4/9, the minimum value in a square cavity.

Moreover, as it was shown for prefractal cavities, the tendency to localisation is increased by the irregular character of the boundary geometry. For example, if one considers the geometry shown in Fig. 1(a), where the upper side of the initial square cavity has been replaced by $N = 5$, 10 and 20 wedges, keeping constant the resonator's surface area, one observes that localisation effects are more and more pronounced. The same effect exists if one goes from a regular wedge structure to a random wedge structure at constant surface area [3].

Note however that not all modes are localised. The modes can be split up into two families. A first group, the non-localised modes, is composed of the modes with typically $S_n/S > 0.2$. They are not shown here. The number of such modes is globally independent of the number of wedges.

Then, there is a second group, composed of an increasing number of very localised modes with $S_n/S < 0.2$. As depicted in Fig. 1, these modes are confined near the irregular wall, in 'sub-cavities'. For this reason, we call such phenomenon 'frontier localisation'.

A most remarkable fact is that the total number of modes in the chosen frequency interval increases with the perimeter to surface ratio, a rough measure of the surface 'complexity'. This reflects the behaviour of the classical correction to the Weyl leading term in the density of modes [4] in term of surface irregularity. But it suggests that the anomaly in the density of states (i.e., the correction term) is a direct consequence of localisation.

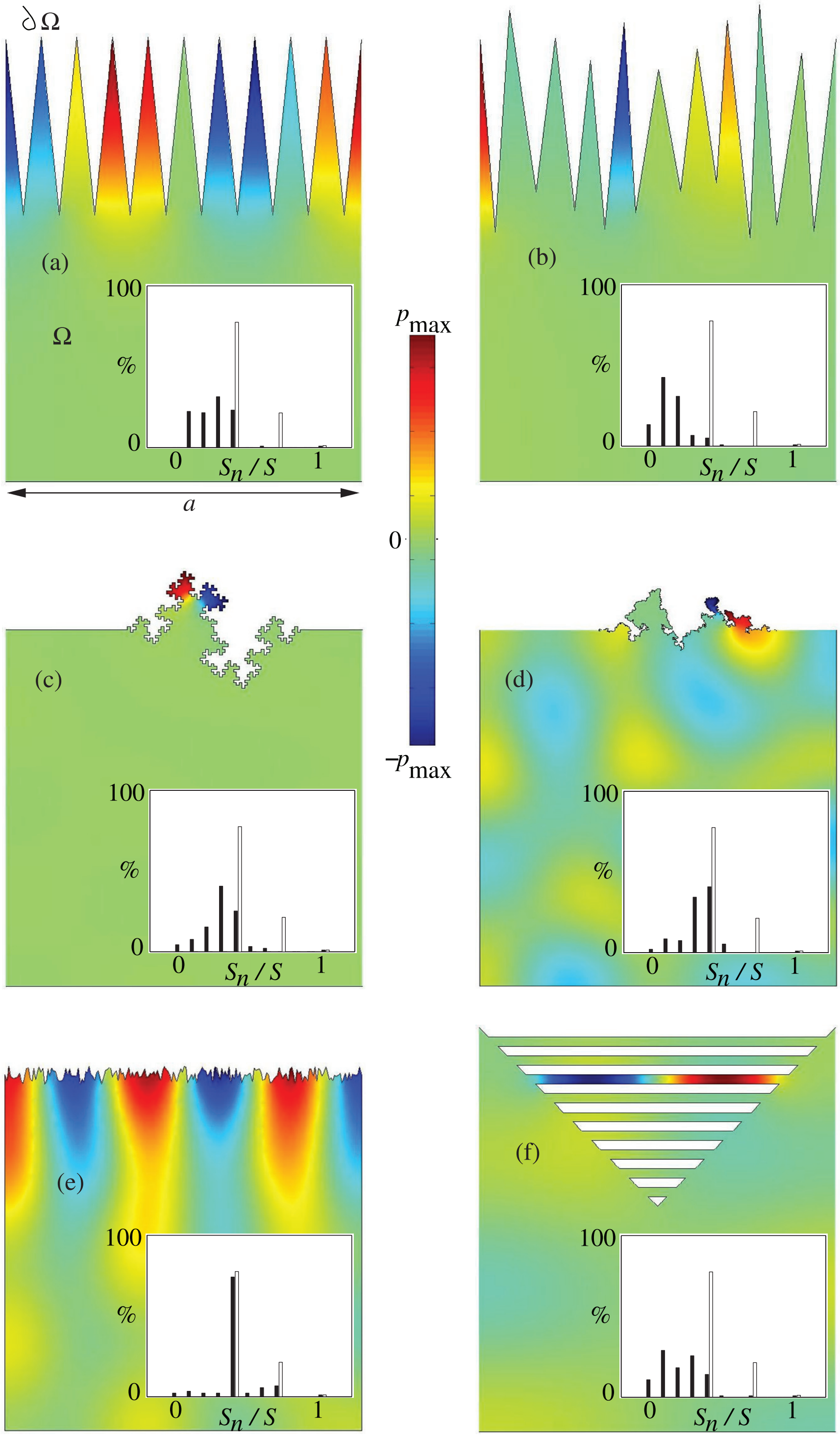

Fig. 1. (Colour online) examples of the amplitude distribution of localised modes in different irregular shallow cavities. One observes that, whatever the type of irregular geometry, there always exist localised modes, i.e., modes which exhibit a very small amplitude in a major fraction of the cavity space. The insets show for each geometry, the histogram of the relative existence surface of the eigenmodes in the frequency range $k_n a/\pi \in [0, 10]$, compared to the results for a square cavity indicated by the empty bars.

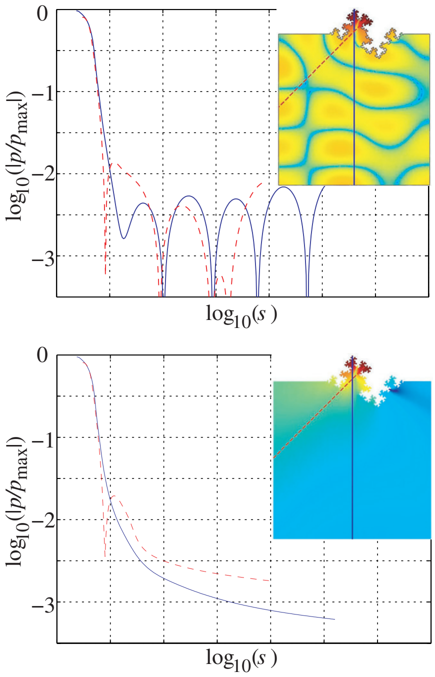

Fig. 2. (Colour online) spatial decrease of the amplitude of a localised mode in two cases. Top: the mode in a cavity with a prefractal defect (Fig. 1(c)). Bottom: the same mode in an open space limited by the same prefractal frontier, and appearing with almost the same eigenfrequency. The red dashed curved gives the amplitude along the corresponding red line on inset pictures, and similarly for the blue solid curves. s is the arc length along these lines.

4 Mechanism of localisation

One can conclude from the above results, that any irregularity with a given size generates one or several localised modes with a wavelength commensurate with that size. An example is the mode shown in Fig. 1(c). But more subtle cases exist, such as the mode shown in Fig. 1(d) which do not correspond to a visible geometric feature, or Fig. 1(e).

The existence of localised modes raises the question of the physical mechanism responsible for this phenomenon. First note that, in contrast with fractal drums in which some of the modes are confined by the Dirichlet boundary condition which forbids propagation in narrow regions, here there is no such "confining force". Here the localisation effects result from spatial decoherence effects, or, in other words, from the fact that diffraction create constructive interferences only in a small fraction of the resonator volume. The interferences are essentially destructive elsewhere [5].

In contrast to strong localisation and exponential decay, the localisation here is weak, characterised by a much slower decrease in space. Our case is displayed in Fig. 2. The top figure illustrates the spatial dependence of a localised state in a resonator. The bottom figure shows the same state in "open space" with non-reflecting boundary conditions on the three sides of the square replacing the Neumann conditions (perfectly matched layers [6] were implemented in FEMLAB for the numerical computations). Figure 2 shows a rapid decrease of the amplitude in the vicinity of the zone of confinement and a slower decrease (much slower than exponential) at larger distance from the zone of confinement. Such a slow decay is characteristic for weak localisation. Let us consider a wave at the frequency of a localised mode, which is coming from the regularly shaped region in the lower part of Fig. 2. It is important to understand the interaction of this wave with the complex structured boundary. The wave will be quite strongly coupled to the slowly decaying localised mode shown in Fig. 2 (top). If such a mode is strongly dissipative the wave will be strongly absorbed as shown below. This indicates that, in general, irregular structures might be good candidates for absorbing waves.

5 Wall dissipation in irregular acoustical cavities

We study the case where the medium filling the cavity is supposed to be lossless (a good approximation for air at audio-frequencies) and we consider the cases of weak losses occurring at the walls. The wall presents a small, but non zero, acoustic admittance $\epsilon(\omega) \in \mathbb{C}$. If $p_n = P_n S^{1/2}\psi_n$ is the pressure field corresponding to the normalised eigenmode ψ_n, the associated dissipated energy W_n, that is the total outflow through the cavity walls, is

$$W_n = \int_{\partial\Omega} \frac{|p_n|^2}{2\rho_0 c_0} \mathrm{Re}(\epsilon)\,\mathrm{d}L, \tag{5}$$

or

$$W_n = \frac{P_n^2 S}{2\rho_0 c_0} \mathrm{Re}(\epsilon) \int_{\partial\Omega} |\psi_n|^2\,\mathrm{d}L, \tag{6}$$

with ρ_0 and c_0 the density and sound velocity in the medium. Then, the energy dissipation is a direct function of the amplitude distribution of the mode along the cavity perimeter by the term

$$w_n = \int_{\partial\Omega} |\psi_n|^2\,\mathrm{d}L. \tag{7}$$

For the normalised modes that we consider, the average square amplitude $|\psi_0|^2$ is of order $1/S_n$. Then the above integral is roughly proportional to the ratio of the perimeter of the existence region divided by the existence surface. This ratio is roughly proportional to the square root of the inverse existence surface. The inverse surface is larger for localised states than for non-localised states. This indicates that localised modes are more strongly damped than non-localised modes. In consequence, irregular structures are good candidates to absorb wave energy.

6 Bulk dissipation in acoustic cavities filled with an irregular absorbent

We now consider how the properties of a cavity filled with an absorbing material depend on the absorbent shape [7–10]. This is the question that has to be solved to understand why electromagnetic [11] anechoic chambers do work better with irregular absorbing walls. From the theoretical point of view, the difficulty lies in the fact that part of the propagation occurs in a lossy material for which the wave operator is non-hermitian.

The only mathematical result concerning this problem has been given in 2d for interfaces that are smooth at the scale of the wavelength. For that case, it was found that the modes are localised either in the non-absorbing region, with approximately real eigenvalues, or are localised in the absorbing region, with eigenvalues having a large imaginary part [12]. The qualitative consequence is that waves excited in the non-absorbing region are, due to localisation, poorly dissipative. Conversely, strongly dissipative modes localised in the dissipative regions are, due to localisation in these regions, poorly coupled to sources in the non-absorbing regions. So, strongly dissipative modes are poorly coupled and strongly coupled modes are poorly dissipative: in both cases the system is poorly absorbing due to localisation.

We show below that strongly coupled and simultaneously dissipative modes occur if one uses an irregular geometry for the interface between the absorbing and the non-absorbing medium. We consider first in some detail, the situation of an acoustic cavity Ω, in which the fields obey simple scalar wave equations. The cavity is rectangular (Fig. 3) but its global shape has no significant effect on the qualitative results that are found. In a region $\Omega_0 \subset \Omega$, the cavity is filled with an homogeneous and lossless medium, while on the complementary part Ω_d of Ω the homogeneous medium is dissipative and possibly dispersive. Let us call Γ the interface, of arbitrary shape, between the two media. Thus, the eigenvalue problem can be written

$$\begin{cases} \Delta\psi_i = -k_i^2\psi_i & \text{in } \Omega_0, \\ \Delta\psi_i = -k_i^2 n^2(\omega)\psi_i & \text{in } \Omega_\mathrm{d}, \end{cases} \tag{8}$$

with the usual continuity conditions on Γ. The quantity $n(\omega) \in \mathbb{C}$ is the complex refraction index of the dissipative medium. For a shallow cavity, the field ψ_i is the acoustic pressure obeying the Neumann condition $\nu \cdot \nabla\psi_i = 0$, where ν is the normal to the surface on the outer boundary. The lossless medium is characterised by a relative density ρ_0 and a relative

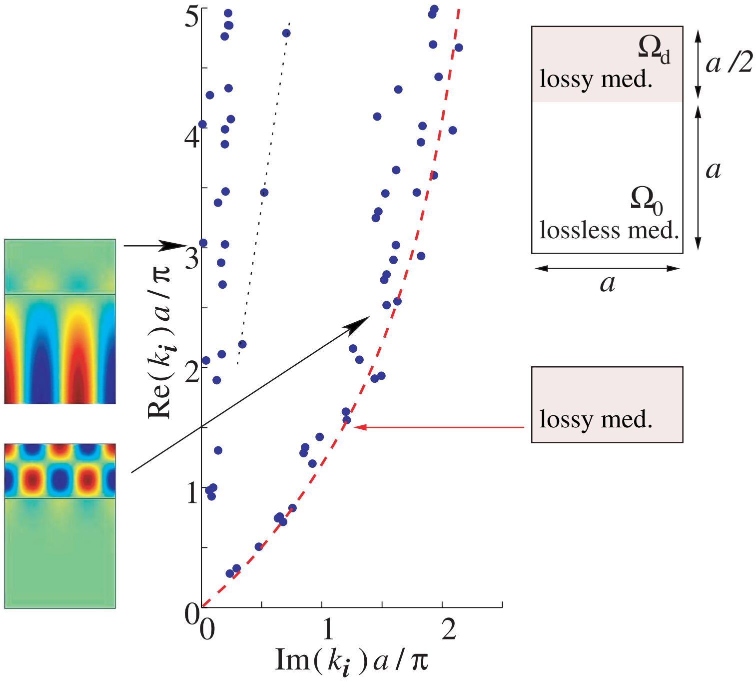

Fig. 3. (Colour online) low frequency complex eigenvalues for the cavity with a planar absorbent material as shown on the top right. The dashed red curve shows the location of the theoretical eigenvalues of a single cavity filled with the same lossy medium. The dotted line shows some of the classical "interface" modes that may arise on a plane interface.

compressibility K_0 both equal to 1, while the corresponding values in the lossy medium are two functions of the frequency, modelling a porous material in the *equivalent fluid* assumption [13]. One should note, however, that the dispersive character of the lossy medium is not a prerequisite to observe the localisation effects in which we are interested.

When the interface between the two media in the cavity is planar, the spectrum of eigenvalues k_i is distributed in the complex plane as shown in Fig. 3. In this figure, one clearly distinguish two families of modes, each following approximately a simple theoretical limit. The first group have nearly real eigenvalues. As indicated in the figure, the corresponding modes "live" mostly in the lossless region of the cavity, decreasing rapidly in the lossy region. The second group is composed of eigenmodes living in the lossy region of the cavity (see Fig. 3) and the corresponding eigenvalues are concentrated along the dashed curve. This curve corresponds to the location of the eigenvalues for an isolated cavity filled with the lossy medium. In this case the eigenvalues can be easily calculated analytically. Indeed, consider a shallow cavity $\mathcal{C}$ of arbitrary shape, filled with a medium having a refraction index $n(\omega) \in \mathbb{C}$, with homogeneous Neumann condition on the boundary. Solving the eigenvalue equation $\Delta\psi_i = -k_i^2 n^2(\omega)\psi_i$ in $\mathcal{C}$ leads to a set of discrete *real* values $\delta_i = k_i n$. Writing that δ_i is real implies that, in the complex plane $(\mathrm{Im}(k), \mathrm{Re}(k))$ the eigenvalues k_i are located on the curve $(-\mathrm{Re}(k)\mathrm{Im}(n)/\mathrm{Re}(n), \mathrm{Re}(k))$. Thus, the inhomogeneous resonator Ω, with a planar separation between the two subdomains, behaves essentially as two *decoupled* resonators Ω_0 and Ω_d. Note that this in itself is an interesting fact: two media separated by a smooth boundary are only weakly coupled. We are back in the situation mentioned above of dissipative modes poorly coupled and strongly coupled modes poorly dissipative.

We now consider the role played by the irregular character of the geometry of the interface between the two media. By this, we mean that the interface exhibits geometrical irregularities with sizes of the order of the wavelengths under consideration. We use first and second generations of the quadratic Koch curve (fractal dimension equal to 3/2) in order to increase the irregularity while keeping constant the quantity of dissipative material (Fig. 4).

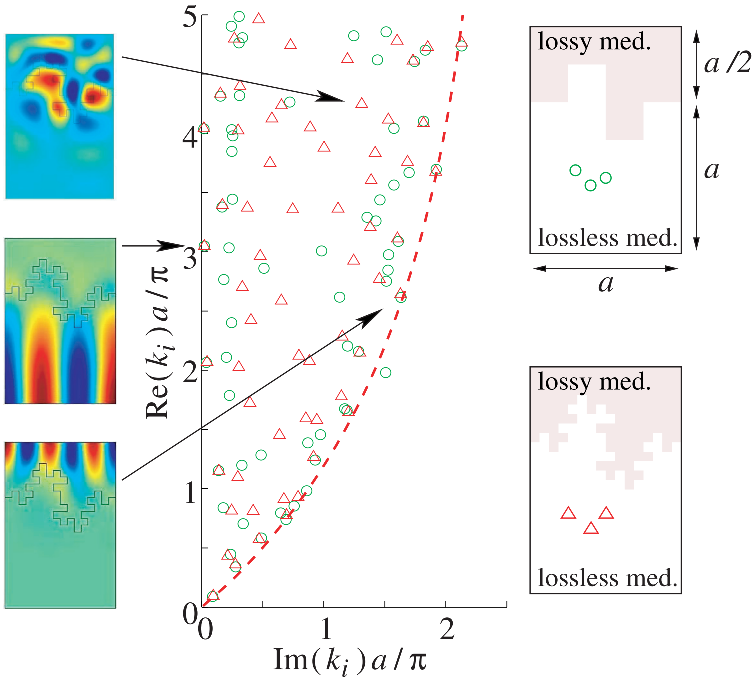

Fig. 4. (Colour online) low frequency complex eigenvalues for the two cavities with irregular absorbers as shown on the right. Circles (triangles) correspond to the first (second) prefractal generation geometry for the interface geometry. The dashed red curve shows the location of the theoretical eigenvalues of a single cavity filled with the lossy medium.

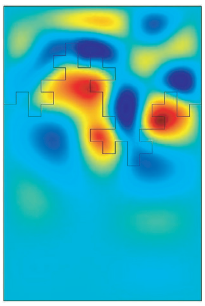

Fig. 5. (Colour online) example of an "astride" mode in a shallow electromagnetic cavity. The lossless medium is characterised by a relative permittivity and permeability both equal to 1, while the corresponding values in the lossy medium are $2-2i$ and 1, so that the refraction index is $n = \sqrt{2-2i}$.

Now, the well discriminated undamped and damped eigenmodes found for the planar interface are replaced by an ensemble of modes with eigenvalues widely spread in the complex plane. The eigenvalues occupy a region delimited by the two theoretical limits described above. There are still a few values close to the former theoretical limits, but most of the modes occupy the space between these limits and the number of these modes increases with the irregularity of the interface.

A major characteristic is that the amplitude distribution of the modes corresponding to these intermediate eigenvalues are located "around" the irregular interface. An example is shown in Fig. 4. We call such modes "astride" modes and the number of these astride modes increases with increasing irregularity.

It is worth to note that these astride modes also exist in shallow electromagnetic cavities. In this case the field is the "vertical" electric field. It now obeys Dirichlet boundary conditions $\psi_i = 0$ on the external boundary of the cavity instead of Neumann boundary conditions for the above acoustic pressure field. An example of an astride electromagnetic eigenmode is shown in Fig. 5.

7 Conclusions

We have shown that resonators with complex geometry always display some kind of localisation effects in the sense that the energy of many eigenmodes is concentrated in a restricted fraction of the volume of the resonator. Two different types of geometrical complexity have been studied: complex shape of the cavity and complex shape of an absorbing material in a cavity of regular geometry.

In the first case, the localised energy is always concentrated along the irregular boundary. Localisation is the manifestation of decoherence effects in a major fraction of the volume and constructive interference in a restricted fraction of the resonator volume. One observes a qualitative link between the number of localised modes, the anomaly of the density of states and the irregular character of the geometry.

The properties of resonators partially filled with an absorbing material strongly depend on the morphology of the interface separating the absorbing from the non-absorbing regions. For simple planar interfaces the modes split in two groups, localised in the absorbing and the non-absorbing regions, respectively. The situation is very different for irregular interfaces. In that case, most of the modes reach both media, 'astride' the irregular interface. These facts are discussed here for a specific prefractal morphology. The same results are obtained for other geometrical irregularities. Since they exhibit simultaneously a significant imaginary part of the eigenvalue and a large amplitude in the non-absorbing region, the astride modes are thought to be important for the absorption of wave energy emitted in the non-absorbing region. We believe that this constitutes the first simple rationale towards understanding the role of the absorbent geometry in anechoic chambers, both, electromagnetic or acoustic. In acoustics, this also implies that the use (and theory) of reverberation chambers to measure the absorption of irregular panels should be reconsidered in order to properly take into account the role of localisation.

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