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Low-complexity computation of plate eigenmodes with Vekua approximations and the Method of Particular Solutions

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Abstract This paper extends the Method of Particular Solutions (MPS) to the computation of eigenfrequencies and eigenmodes of plates. Specific approximation schemes are developed, with plane waves (MPS-PW) or Fourier-Bessel functions (MPS-FB). This framework also requires a suitable formulation of the boundary conditions. Numerical tests, on two plates with various boundary conditions, demonstrate that the proposed approach provides competitive results with standard numerical schemes such as the Finite Element Method, at reduced complexity, and with large flexibility in the implementation choices.

Keywords element-free methods · biharmonic equation · numerical methods · algorithms · eigenvalues

1 Introduction

Numerical computation of eigenfrequencies and eigenmodes of plates is an important problem in mechanics. An eigenmode is a non-zero solution to

$$D\Delta^2 u + T\Delta w - \rho h \omega^2 u = 0 \quad (1)$$

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with boundary conditions, where D is the rigidity of the plate, ρ the specific mass of its material, h its thickness, and T the normal tension applied at its edges, assumed to be uniform. The eigenfrequencies are the frequencies ω such that a non-zero solution exists. Apart from particular cases where these quantities can be analytically computed (e.g., circular plates with simple boundary conditions), they must be obtained by numerical methods.

The finite element method (FEM), which uses piecewise polynomials to approximate the solutions, can be used to compute these eigenmodes and eigenfrequencies. However, it can be computationally intensive at high frequencies, as the size of the numerical problem scales as the square of the spatial frequency. Alternative methods are the boundary element method (BEM) [1], the method of fundamental solution (MFS) [2], or its variant proposed by Kang *et al.* [3], the Non Dimensional Influence Function (NDIF). Here, only the solutions to the equation with a given wave number are approximated as linear combinations of a family of functions. Eigenmodes are found as such combinations which also satisfy the boundary conditions.

Here, we investigate a new computational method derived from the Method of Particular Solutions, proposed by Fox, Henrici and Moller (FHM) [4], and improved by Betcke and Trefethen [5], for the computation of eigenmodes of the Laplace operator. While the fundamental ideas used in this method are similar to the previously cited methods, it has several advantages:

- the stability of the numerical problems is improved,
- multiple eigenvalues are easier to determine,
- and basis of the associated eigenspaces are readily available.

The goal of this paper is to extend this MPS framework, presented in section 2, to the computation of eigenmodes and eigenvalues of plates. Because of fundamental limitations of the Vekua theory, we apply the method only on star-shaped plates with smooth boundaries. Our first contribution is the analysis of an approximation scheme based on the Vekua theory, given in section 3. This provides some bounds on the approximation error of a solution of eqn. (1) by sums of Fourier-Bessel functions in Sobolev norms, based on similar results for the simpler case of the Helmholtz equation. Our second contribution, described in section 4, is the formulation of the problem in a way compatible with the MPS, and its numerical evaluation presented in section 5. We discuss the extension to more general shapes, and various implementation matters, in section 6.

2 The method of particular solutions

The method of particular solutions was introduced by Fox, Henrici and Moller (FHM) [4] for the case of eigenmodes of the Laplace operator in a L-shaped domain with a singular corner.

The basic idea of this method is, instead of considering the entire space in which the eigenmodes are searched (e.g. H^1 for the case of the Laplace operator, approximated by finite element spaces), to consider separately the spaces of solutions of the Helmholtz equation for different wave numbers. Then, in each of these spaces, we can look for a nonzero function which also satisfies the boundary conditions, *i.e.*, an eigenmode. While building approximations for a lot of different spaces seem to be counter-productive compared to the unique approximation needed for a Galerkin approximation, this alternative scheme is interesting as efficient approximations can be obtained for these spaces of solutions to the Helmholtz equation.

The method developed by FHM, for the computation of eigenmodes of the Laplace operator with Dirichlet boundary conditions, is as follows. For each wavenumber k, we consider N points x_j on the border of the domain, and a family of N functions ϕ_i spanning a subspace which approximates the set of solutions to the Helmholtz equation. The considered family, Fourier-Bessel functions of fractional orders, was specifically constructed to take into account the singularity arising in the reentrant corner of the L-shaped domain. In order to find the eigenfrequencies, one has to construct a square matrix $M(k)$ that contains the values of the functions ϕ_j at the N points of the border, and to compute its determinant $d(k)$:

$$d(k) = \det M(k) = \begin{vmatrix} \phi_1(x_1) & \cdots & \phi_N(x_1) \\ \vdots & & \vdots \\ \phi_1(x_N) & \cdots & \phi_N(x_N) \end{vmatrix} \qquad (2)$$

If k is an eigenfrequency, there is a non-zero solution to the Helmholtz equation with values zero on the boundary, and its approximation $\sum \alpha_i \phi_i$ is thus close to zero at the sampling points, with nonzero coefficients α_i. The image of the vector of coefficients (α_i) by the matrix $M(k)$ is precisely the vector of the values of $\sum \alpha_i \phi_i$ on the points of the border. This means that the determinant of the matrix $M(k)$ is close to zero. Therefore, the eigenfrequencies are obtained as local minima of $d(k)$.

Numerous variants of this method, using different approximation schemes, have been developed since the original article. They mostly differ by the functions used to approximate the solutions: the MFS uses fundamental solutions, the NDIF [3] uses Bessel functions of the first kind of order 0, etc.

As pointed out in [5], this simple method has known limitations. The discretization of the space of solutions and the sampling of the boundaries must be of the same size, and more importantly, the matrix $M(k)$ gets ill-conditioned as the number N of functions grows. Indeed, using a larger family of functions ϕ_i, in order to have better approximations of the modes, makes the problem numerically unstable. An interpretation of this fact is that the algorithm does not search for a non-zero function inside the domain with value zero on its boundary, but actually for a function with non-zero coefficients of its expansion, with zero value on the boundary. The properties of the approximating families are such that having non-zero expansion coefficients does not ensure significant values of the function inside the domain.

To avoid these problems, Betcke and Trefethen [5] suggest to solve, for each k, the following optimization problem :

$$\tau(k) = \min_u \|Tu\|^2_{L_2(\partial\Omega)} \text{ such that } \|u\|^2_{L_2(\Omega)} = 1 \text{ and } \Delta u + k^2 u = 0 \qquad (3)$$

where T is the trace operator on the boundary $\partial\Omega$. Then, k is an eigenfrequency if and only if $\tau(k)$, called the tension, is zero.

This problem can be discretized as follows. A family of functions (ϕ_i) is chosen for the approximation of the solutions of the Helmholtz equation. For a function with expansion coefficients $\mathbf{u} = (u_1, \dots, u_N)$,

$$\|Tu\|^2_{L_2(\partial\Omega)} = \mathbf{u}^\star \mathbf{F} \mathbf{u}$$

$$\|u\|^2_{L_2(\Omega)} = \mathbf{u}^\star \mathbf{G} \mathbf{u}$$

where the coefficients of the matrices $\mathbf{F}$ and $\mathbf{G}$ are

$$F_{ij} = \langle T\phi_i, T\phi_j \rangle_{L_2(\partial\Omega)}, \qquad G_{ij} = \langle \phi_i, \phi_j \rangle_{L_2(\Omega)} \tag{4}$$

These scalar products can be estimated by sampling the domain and its border, and using a Monte-Carlo approximation of the integrals. The optimization problem (3) can be replaced by the generalized eigenvalue problem

$$\lambda \mathbf{F} \mathbf{u} = \mathbf{G} \mathbf{u} \tag{5}$$

for which the largest eigenvalue is the inverse of $\tau(k)$. Note that here, the size of the matrices is the size of the approximating family, and does not depend on the number of samples used in the domain and on its boundary.

The eigenfrequencies are found as the local minima of $\tau(k)$. This method offers additional advantages. The coefficients of the expansion of the eigenmodes are readily available as the first eigenvector of eqn. (5), and n-multiple eigenfrequencies are characterized by the fact that the n first eigenvalues of eqn. (5) exhibit a local minima. Here, a basis of the eigenspace is obtained using the n first eigenvectors of eqn. (5). Note that this basis has no reason to be orthogonal, as it is the vectors of the coefficients of the expansions of the basis functions that are orthogonal, not the functions themselves.

The same method, with a slightly different implementation, was used by Barnett and Berry [6] to compute high frequency modes of quantum cavities. As they considered only domains with smooth boundaries, plane wave families were sufficient for the application of the method.

To generalize this method to plates, two adaptations are needed:

- an approximation scheme for solutions of eqn. (1) has to be developed
- the boundary conditions encountered in plate problems have to be modeled in a way compatible with formulation eqn. (3).

These two points are the topics of the next two sections, respectively.

3 Approximation of plate eigenmodes

In this section, we prove that solutions of eqn. (1), with arbitrary boundary conditions, can be approximated by sums of Fourier-Bessel functions and modified Fourier-Bessel functions. We first give a short account of the Vekua theory for the Laplace operator, and then extend these results to the bi-Laplace operator.

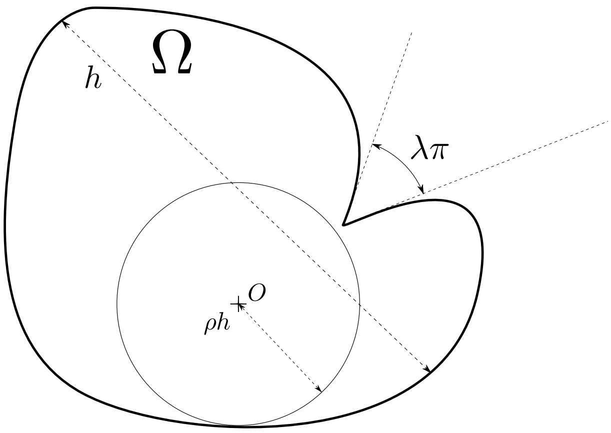

Fig. 1 A star-shaped domain satisfying the exterior cone condition with angle $\lambda\pi$

The domain where the functions are defined is assumed to be star-shaped, and to contain the ball centered on the origin of radius ρh, where h is the diameter of the domain. Furthermore, we assume that the domain satisfies the exterior cone condition with angle $\lambda\pi$. This means that each point of the border is the vertex of a cone of angle $\lambda\pi$ which does not intersect the interior of the domain. Such a domain is pictured on figure 1.

In the following, we give the approximation bounds in weighted Sobolev norms defined by

$$\|u\|^2_{m,k} = \sum_{p=0}^{m} \frac{1}{k^{2p}} \sum_{p_1+p_2=p} \int_\Omega \left| \frac{\partial^p}{\partial x^{p_1} \partial y^{p_2}} u \right|^2 dxdy \tag{6}$$

3.1 Vekua theory for the Laplace operator

A simple example of an approximation of a solution to a differential equation is the case of holomorphic functions. These functions, solutions to the Cauchy-Riemann equations, can be, on a simply connected domain, approximated by polynomials of the complex variable. By taking the real part of an holomorphic function and of its approximations, it is shown that we can approximate an harmonic function, solution to $\Delta u = 0$, by harmonic polynomials in $\mathbf{R}^2$.

The Vekua theory, exposed in [7] and summarized in [8], gives similar results for solutions of elliptic PDEs by generalizing the operation "taking the real part", which allows us to map holomorphic functions to harmonic functions, to solutions of these PDEs. Using these operators, which are continuous and continuously invertible, approximation of holomorphic functions by polynomials of the complex variable is translated to approximation of solutions to the PDEs by the images of polynomials.

In the case of the Helmholtz equation

$$\Delta u + \lambda u = 0 \quad (7)$$

Further details on these operators and their properties in Sobolev spaces can be found in [9].

The main result of Moiola *et al.* is that the solutions of eqn. (7) can be approximated by generalized harmonic polynomials, *i.e.* functions of the form

$$u_L = \sum_{n=-L}^{L} \alpha_n J_n(kr) e^{in\theta} \quad (8)$$

where (r, θ) are the polar coordinates.

When λ is strictly positive, Moiola *et al.* [9] have shown that there exists a generalized harmonic polynomial of order at most L such that

$$\|u - Q_L\|_{j,k} \leq C(1 + (kh)^{j+6}) e^{3(1-\rho)kh/4} \left(\frac{\log(L+2)}{L+2}\right)^{\lambda(K+1-j)} \|u\|_{K+1,k} \quad (9)$$

In the case where $\lambda < 0$, the bound is multiplied by $e^{3kh/2}$.

When the function to be approximated is infinitely differentiable in an open domain containing Ω, the convergence is exponential in L [10].

Moiola *et al.* used this results to analyze the approximation of solutions of eqn. (7) by sums of plane waves [11]. The orders of convergence are identical to the generalized harmonic polynomials case.

3.2 Extension to plates eigenmodes

Eigenmodes of an homogenous plate of rigidity D, specific mass ρ and thickness h, with in-plane tension T, are solutions of

$$D\Delta^2 w + T\Delta w - \rho h \omega^2 w = 0. \quad (10)$$

In order to approximate such functions, we reduce this problem to the approximation of two solutions of the Helmholtz equation. Indeed, solutions of equation (10) can be decomposed as a sum of two solutions of equation (7), with parameters deduced from the properties of the plate.

Lemma 1 *Let w a solution of (10) in the sense of distributions. Then w can be decomposed as the sum of w_1 solution of $\Delta w_1 - \lambda_1 w_1 = 0$, and w_2 solution of $\Delta w_2 - \lambda_2 w_2 = 0$, where λ_1 and λ_2 are the zeros of*

$$D\lambda^2 + T\lambda - \rho h \omega^2. \quad (11)$$

Furthermore, if $w \in H^{K+2}$, then

$$||w_1||_{K,k_1} \leq \frac{2k_+^2}{\delta_\lambda} ||w||_{K+2,k_1} \quad (12)$$

$$||w_2||_{K,k_2} \leq \frac{2k_+^2}{\delta_\lambda} ||w||_{K+2,k_2}. \quad (13)$$

where $k_1 = \sqrt{|\lambda_1|}$ and $k_2 = \sqrt{|\lambda_2|}$, k_+ being the largest, and $\delta_\lambda = \sqrt{T^2 + 4D\rho h\omega^2}/D$ the difference between λ_1 and λ_2

Proof Analysis : assuming the decomposition, we find

$$\Delta w - \lambda_2 w = (\Delta w_1 - \lambda_2 w_1) + (\Delta w_2 - \lambda_2 w_2) \quad (14)$$
$$= (\lambda_1 - \lambda_2) w_1 + 0 \quad (15)$$

A similar computation for w_2 gives

$$w_1 = \frac{1}{\lambda_1 - \lambda_2}(\Delta w - \lambda_2 w), \quad w_2 = \frac{1}{\lambda_2 - \lambda_1}(\Delta w - \lambda_1 w).$$

Note that λ_1 and λ_2 are always distinct as δ_λ is always strictly positive.

Synthesis : we check that $w = w_1 + w_2$:

$$w_1 + w_2 = \frac{1}{\lambda_1 - \lambda_2}(\Delta w - \lambda_2 w) - (\Delta w - \lambda_1 w) \quad (16)$$
$$= w \quad (17)$$

Then that $\Delta w_1 - \lambda_1 w_1 = 0$:

$$\Delta w_1 - \lambda_1 w_1 = \frac{1}{\lambda_1 - \lambda_2}\left((\Delta^2 w - \lambda_2 \Delta w) - (\lambda_1 \Delta w - \lambda_1 \lambda_2 w)\right) \quad (18)$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left(\Delta^2 w - (\lambda_2 + \lambda_1)\Delta w + \lambda_1 \lambda_2 w\right) \quad (19)$$

$$= 0 \quad (20)$$

The last equality comes from the fact that λ_1 and λ_2 are the zeros of the polynomial $D\lambda^2 + T\lambda - \rho h\omega^2$. We also find that $\Delta w_2 - \lambda_2 w_2 = 0$.

Finally, if $w \in H^{K+2}$, then

$$\|w_1\|_{K,k_1} \leq \frac{1}{|\lambda_1 - \lambda_2|}\left(\|\Delta w\|_{K,k_1} + \lambda_2 \|w\|_{K,k_1}\right) \quad (21)$$

$$\leq \frac{1}{\delta_\lambda}\left(k_1^2 \|w\|_{K+2,k_1} + k_2^2 \|w\|_{K+2,k_1}\right) \quad (22)$$

$$\leq \frac{2k_+^2}{\delta_\lambda} \|w\|_{K+2,k_1} \quad (23)$$

The result is identical for w_2.

Remark 2 *In the case where no tension is applied to the plate, i.e. $T = 0$, we have $k_1 = k_2 = (\rho h/D)^{1/4}\sqrt{\omega} = k$, $\delta_\lambda = 2k^2$ and the results can be simplified as :*

$$||w_1||_{K,k} \leq ||w||_{K+2,k}, \quad ||w_2||_{K,k} \leq ||w||_{K+2,k}.$$

Remark 3 *Two orders are lost in the majorizations* (12) *and* (13) *: the norm of order K of both components of w are bounded by their norm of order $K+2$. It is impossible, in the general case, to have a better bound. Let consider, on a disk sector centered at the origin, the function f defined by $f = e^{i\theta/2}(J_{1/2}(r) - I_{1/2}(r))$ in polar coordinates. This function is solution of $\Delta^2 w - w = 0$, and can be decomposed as the sum of $f_1 = e^{i\theta/2}J_{1/2}(r)$, solution of $\Delta f_1 + f_1 = 0$, and $f_2 = -e^{i\theta/2}I_{1/2}(r)$, solution of $\Delta f_2 - f_2 = 0$. These two functions have a radial behavior at the origin similar to $r^{1/2}$, and thus are not in H^3. Their sum however behaves like $r^{5/2}$, and is in H^4.*

If λ_1, or λ_2, is negative, the corresponding component of w can be readily approximated by generalized harmonic polynomials or plane waves using the results of Moiola *et al.* [11]. If λ_1 or λ_2 is positive, the associated component can be approximated in a similar way. In that case, we use either a family of modified Fourier-Bessel functions; where the Bessel functions J_n are replaced by modified Bessel functions I_n, or a family of exponential functions $e^{\mathbf{k}\cdot\mathbf{x}}$ instead of plane waves. The bounds on the approximation error are similar.

Theorem 4 *Let Ω be a domain satisfying the assumptions of this section, $K \geq 1$ integer, and $w \in H^{K+2}$ verifying conditions of lemma 1. Then for all $L > K$, there exist two generalized harmonic polynomials P_L and Q_L with parameters λ_1 and λ_2, of degree at most L such that for all $j \leq K$,*

$$\|w - (P_L + Q_L)\|_{j,k_+} \leq C\frac{k_+^2}{\delta_\lambda}(1 + (k_+h)^{j+6})e^{\frac{3}{4}(3-\rho)k_+h} \left(\frac{\ln(L+2)}{L+2}\right)^{\lambda(K-j)} (k_+h)^{K-j}\|w\|_{K+2,k_-}, \tag{24}$$

where k_- is the smallest of k_1 and k_2, and k_+ the largest. If λ_1 and λ_2 are both negative, the bound can be divided by $e^{3/2k_+h}$.

Proof The first step of the proof is to decompose w using lemma 1. Using theorem 2.2.1.ii and remark 1.2.6 from [12], we can approximate these two components by generalized harmonic polynomials, modified if λ is positive.

Let us assume $\lambda_1 < 0$, $\lambda_2 > 0$, with $|\lambda_2| > |\lambda_1|$. Other cases can be treated similarly.

We have

$$\begin{aligned}\|w - (P_L + Q_L)\|_{j,k_+} &\leq \|w_1\|_{j,k_+} + \|w_2\|_{j,k_+} \\ &\leq \|w_1\|_{j,k_1} + \|w_2\|_{j,k_2} \\ &\leq C(1 + (k_1h)^{j+6})e^{\frac{3}{4}(1-\rho)k_1h} \\ &\left(\frac{\ln(L+2)}{L+2}\right)^{\lambda(K-j)} (k_1h)^{K-j}\|w_1\|_{K,k_1} \\ &+ C(1 + (k_2h)^{j+6})e^{\frac{3}{4}(3-\rho)k_2h} \\ &\left(\frac{\ln(L+2)}{L+2}\right)^{\lambda(K-j)} (k_2h)^{K-j}\|w_1\|_{K,k_2} \\ &\leq C\frac{k_+^2}{\delta_\lambda}(1 + (k_+h)^{j+6})e^{\frac{3}{4}(3-\rho)k_+h} \\ &\left(\frac{\ln(L+2)}{L+2}\right)^{\lambda(K-j)} (k_+h)^{K-j}\|w\|_{K+2,k_-}\end{aligned}$$

Remark 5 *For plates without in-plane tension, the result is slightly simpler. In that case, the roots of eqn. (11) have same absolute values and opposite signs. We thus have*

$$\|w - (P_L + Q_L)\|_{j,k} \leq C(1 + (kh)^{j+6})e^{\frac{3}{4}(3-\rho)kh} \left(\frac{\ln(L+2)}{L+2}\right)^{\lambda(K-j)} (kh)^{K-j}\|w\|_{K+2,k}. \tag{25}$$

Remark 6 *We prove the approximation result for sums of Fourier-Bessel functions. However, as Fourier-Bessel functions can be approximated by sums of plane waves, modified Fourier-Bessel functions can be approximated by sums of exponential function of the type $e^{\mathbf{k}\cdot\mathbf{x}}$ with constant norm of $\mathbf{k}$. This allows an approximation of solutions of eqn.* (10) *by sums of plane waves and exponential functions.*

4 Boundary conditions

Boundary conditions usually encountered in plate problems (clamped edges, simply supported edges and free edges) are not as such readily usable for the numerical scheme proposed here, and thus need to be modeled differently.

For clamped edges, the displacement and its normal derivative are zero. The tension for clamped boundary conditions is then:

$$t_c = \int_\Gamma |w|^2 + \frac{1}{k^2}\int_\Gamma \left|\frac{\partial w}{\partial n}\right|^2.$$

A simply supported edge has zero displacement and torsion moment M_n

$$t_s = \int_\Gamma |w|^2 + \frac{1}{D^2k^4}\int_\Gamma |M_n|^2.$$

Finally, free edges have zero torsion moment and Kelvin-Kirchhoff edge reaction K_n

$$t_l = \frac{1}{D^2k^4}\int_\Gamma |M_n|^2 + \frac{1}{D^2k^6}\int_\Gamma |K_n|^2 .$$

The torsion moment and the Kelvin-Kirchhoff edge reaction writes [13]

$$M_n = -D\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right)$$

$$K_n = -D\left(\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n\partial t^2} + \frac{1-\nu}{R}\left(\frac{\partial^2 w}{\partial n^2} - \frac{\partial^2 w}{\partial t^2}\right)\right)$$

where ν is the Poisson ratio of the material, and R the curvature radius of the boundary.

The constants $1/k^2$, $1/D^2k^4$ and $1/D^2k^6$ in front of some integrals not only make the quantities homogeneous, but also improve the numerical stability as the contribution of the two integrals are rescaled to have the same order of magnitude. For instance, in the case of t_c estimated using plane waves, the first term contains products of planes waves, while the second term contains products of derivatives of plane waves, which are the plane waves themselves multiplied by the scalar product of the wave vector and a unit vector normal to the boundary. With no rescaling, the second term would be more and more influent as the frequency increases. This rescaling is similar to the weights used to define the norms $\|\cdot\|_{m,k}$ in eqn. (6).

Note that for a plate with various boundary conditions along the border, the corresponding tension is the sum of the tension for the different boundary conditions, integrated on their respective domain.

5 Numerical results

We now compare the methods with analytical results for simple cases, and with numerical results obtained with Cast3M [14], a widely-used FEM simulation program. To avoid technicalities, the simulated plates are star-shaped with smooth boundaries. The treatment of more general shapes is discussed in the next section. Since these numerical tests are done without in-plane tension, we can use the wavenumber $k = (\rho/D)^{1/4}\sqrt{\omega}$ to express the eigenfrequencies. In the case of in-plane tension, the wavenumbers used to generate the Fourier-Bessel functions (resp. plane waves) and modified Fourier-Bessel functions (resp. exponential functions) are computed according to lemma 1.

We first test the method on a circular plate of radius 1, with various boundary conditions. Here, we use plane waves, as eigenmodes of circular plates are sums of a Fourier-Bessel function and a modified Fourier-Bessel function, making the numerical problem trivial. Table 1 gives the computed eigenfrequencies and the values given in Leissa [15] for low-frequency modes. Figure 2 shows the tension as a function of the frequency in the clamped case, and the next three eigenvalues of the numerical problem (5). These can be used to identify multiple eigenvalues and to compute a basis of the associated eigenspaces. Examples of eigenmodes are given for the three boundary conditions on figure 3. In these numerical experiments, the boundary of the disk was discretized with 2048 points, its interior with 1024 points drawn using the uniform distribution on Ω, and the number of plane waves (which is also the size of the numerical problems to solve) was taken as $8k$, which in this case ranges from 28 to 80.

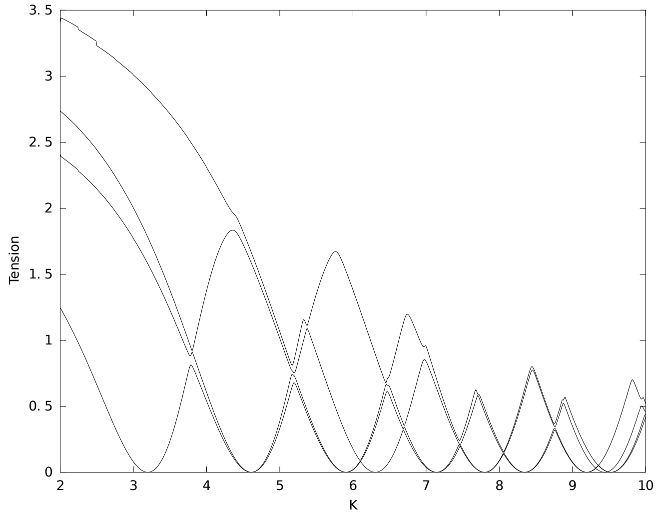

Fig. 2 Inverses of the four largest eigenvalues (the first one being the tension) of discrete problem (5), for the clamped circular plate

	Clamped		Simply supported		Free	
	Leissa	MPS-PW	Leissa	MPS-PW	Leissa	MPS-PW
1	3.196	3.196	2.23	2.22	2.29	2.29
2	4.611	4.611	3.73	3.73	3.01	3.01
3	5.906	5.905	5.06	5.06	3.50	3.50
4	6.306	6.306	5.46	5.45	4.53	4.53
5	7.144	7.144	n/a	6.32	4.65	4.64

Table 1 Low eigenfrequencies for a circular plate with various boundary conditions, in Leissa [15] and computed with the MPS and plane waves (MPS-PW).

We now compute eigenfrequencies and eigenmodes of a second plate, of mode complex shape, with boundaries defined by the parametric equations

$$\begin{cases} x = \cos t \\ y = \sin t + \sin \frac{2t}{3} \end{cases} \qquad t \in [0, 2\pi \tag{26}$$

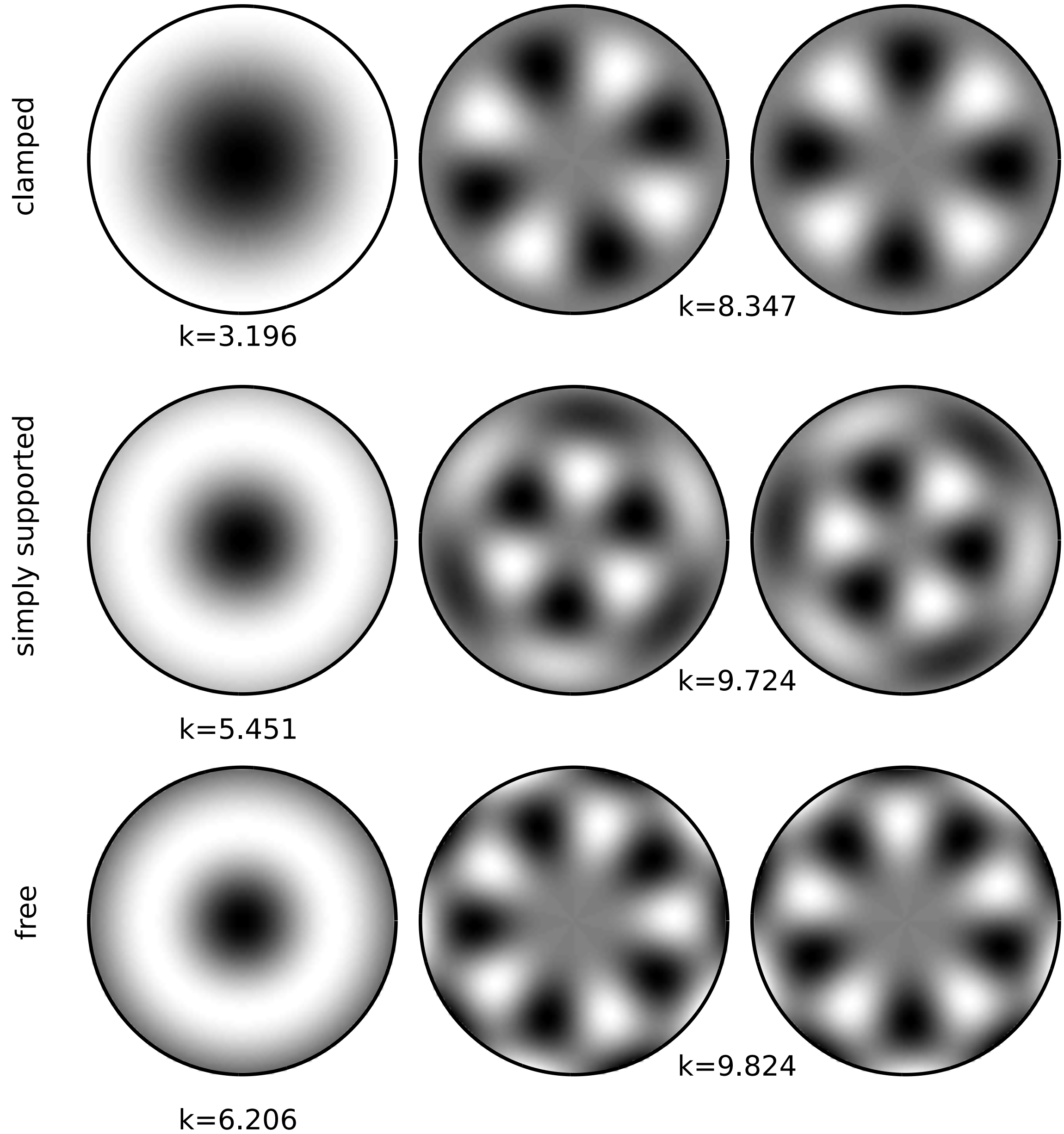

Fig. 3 Some examples of eigenmodes, for simple and double eigenvalues, of the circular plate with clamped, simply supported and free boundary conditions

Table 2 gives the first ten eigenfrequencies of this plate with the three types of boundary conditions : clamped, simply supported and free. The proposed method, MPS with plane waves (MPS-FB), uses 2048 points on the border and 1024 points inside. The number of planes waves is $10k$, which here ranges from 20 to 80. For the clamped conditions, the results using Fourier-Bessel functions are also given (method MPS-FB). We compare our results with those obtained by the FEM, as implemented in the Cast3M package, with 6624 elements and the border discretized by 180 segments.

For two eigenmodes of the clamped plate, we give in table 3 the estimated eigenfrequencies with varying size of the discrete problems for FEM and the MPS-PW, and compare them to the values obtained in [2] with the MFS (with an unspecified size). These two modes, computed with the MPS-PW, are shown on figure 4.

6 Discussion

We now discuss some implementation details, relative to the treatment of more general shapes, or the acceleration of the computations.

	Clamped			Simply supp.		Free	
	FEM	MPS-PW	MPS-FB	FEM	MPS-PW	FEM	MPS-PW
1	3.3498	3.3782	3.3562	2.3819	2.4061	2.2189	2.2000
2	4.5931	4.5944	4.5864	3.6789	3.6733	2.3473	2.4001
3	4.8717	4.9275	4.8815	3.9539	3.9993	2.7643	2.7481
4	5.8588	5.8376	5.8366	4.9799	4.9505	3.4544	3.4222
5	6.1655	6.2067	6.1987	5.2874	5.3216	3.7476	3.7623
6	6.3869	6.4567	6.4527	5.4832	5.5456	4.1207	4.0993
7	7.1179	7.0738	7.0738	6.2658	6.2147	4.2170	4.2004
8	7.4596	7.4949	7.4949	6.5912	6.6188	4.8551	4.7985
9	7.7190	7.7740	7.7590	6.8597	6.9058	4.9701	4.9845
10	7.8910	7.9300	7.9660	6.9979	7.0308	5.4432	5.4316

Table 2 First ten eigenfrequencies, expressed in wavenumber k, of the second plate with various boundary conditions, for FEM, MPS with plane waves (MPS-PW) and MPS with Fourier-Bessel functions (MPS-FB, clamped conditions only)

MFS	FEM				MPS-PW		
	number of elements				number of plane waves		
	246	1018	6624	26966	50	60	70
9.11259	8.9329	9.0245	9.0590	9.0637	9.1068	9.1121	9.1126
9.27903	9.1025	9.1830	9.2122	9.2165	9.2651	9.2722	9.2787

Table 3 Eigenfrequencies for two modes of the clamped second plate, with various discretization sizes, computed by FEM (as implemented in Cast3M) and MPS-PW. Comparison with MFS [2]

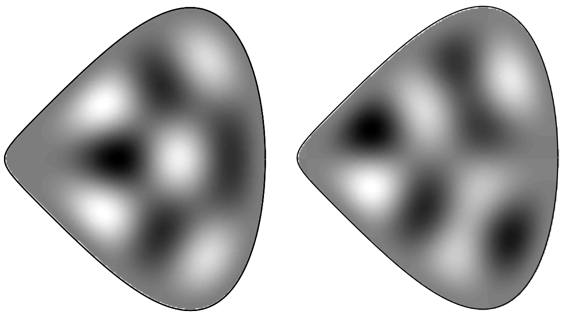

Fig. 4 Eigenmodes of the second plate, for $k = 9.1126$ (left) and $k = 9.2787$ (right) computed by MFS-PW, for the clamped boundary conditions

6.1 Fourier-Bessel vs. plane waves

As shown by the numerical experiments, both Fourier-Bessel functions and plane waves can be used to approximate eigenmodes. They have similar approximation properties, but differ implementation-wise. Fourier-Bessel functions are orthogonal on a disc, ensuring better stability, while plane waves are more and more ill-conditioned as their number increases. However, this can be treated by pre-conditioning the plane waves family with a discrete Fourier Transform, mapping the plane waves to approximations of the Fourier-Bessel functions.

The main advantage of the plane waves is the straightforward computation of their derivatives : differentiating a plane wave along a certain direction amounts to multiplying it with the scalar product of its wave vector with a unit vector. This makes the construction of the matrices both easy to implement and fast.

6.2 Shapes

The proposed method relies on an approximation scheme for solutions to the equation (1), and therefore is limited to cases where such approximations are available. In particular, the approximation scheme assumes a star-shaped domain. In the case of a simply connected, but non-star convex, domain, the approximation is not guaranteed to succeed. A possible way to overcome the problem is to cut the domain into star-convex subdomains, to approximate the solutions of (1) in these subdomain, and to add terms in the tension ensuring that the displacement, normal derivative of the displacement, bending moment and strain have the same value at both sides of the internal boundaries. A similar method as already been applied to the particular case of polygonal membranes [16].

Other cases of non-convex domains are domains with holes, but star-convex if the holes are filled. In that case, following Vekua, one can approximate solutions of (1) by adding to the family of planes waves or Fourier-Bessel functions, the sets of Fourier-Bessel functions of second kind $e^{in\theta}Y_n(kr)$ and $e^{in\theta}K_n(kr)$, one for each hole, centered on a point chosen in each of them. This type of approximation can be compared to the approximation of holomorphic functions given by the Runge theorem. An application to membranes can be found in [17].

6.3 Singularities

The domains considered here have smooth boundaries. This guarantees that the eigenmodes of the plates are smooth, and that the convergence of the approximations is fast. However, as shown in theorem 4, singularities, which can appears at corners of a domain with non-smooth boundaries, slow down the convergence of the approximations. The size of the discretization for such cases is then larger than for smooth boundaries, possibly too large to guarantee the numerical stability of the computations. To accelerate the convergence, FHM and BT use fractional Fourier-Bessel functions centered on the singular corner of the domain (in their numerical experiments, an L-shaped polygon). This idea as been used for plates in a different setting by De Smet et al. [18].

Note that, for a plate with polygonal holes, at least a corner of the hole is singular. In that case, it is impossible to use fractional Fourier-Bessel functions, as it is impossible to define them on a domain containing a path around the origin. In that case, one can combine fractional Fourier-Bessel function with the method described in the previous section [17].

6.4 Numerical considerations

Although the numerical stability is improved compared to the determinant based MPS, the improved version is still prone to instabilities when a large approximation order is used. This instabilities are further amplified in the case of plates, where modified Fourier-Bessel functions, or exponential functions, are included in the approximating families. The behavior of these functions are such that they are non-negligible only on a small region near the boundary of the domain. Using the Monte-Carlo approximation with uniform density to estimate the coefficients of the matrices is thus unstable. Using a non-uniform density of samples, with more samples near the border would be a way to improve the stability of the estimations, in a way similar to what is used in [19] , where the reconstruction of a solution of the Helmholtz equation on a disc is improved by placing a fraction of the samples on the border of the disc.

6.5 Speeding up the eigenvalue search

In order to locate the minima of the tension, the proposed algorithm simply computed it on linearly spaced values in the interval we were interested in. However, the particular behavior of the tension (and more generally, of the eigenvalues of problem (5)) could be used to accelerate the search of these minima. Indeed, the tension is a sequence of branches which behave more or less as parabolas. Newton iterations, along with the computation of the derivatives of the eigenvalues of problem (5), could therefore be used to quickly locate the minima.

7 Conclusion

This paper has described the extension of the Method of Particular Solutions for the computation of eigenmodes of plates. This method has numerous advantages. It can be used with any approximation scheme for the solutions of the studied equation ; in this paper, we used Fourier-Bessel functions and plane waves, but the method could be extended with fractional Fourier-Bessel functions in order to treat singularities. Its formulation also offers a large flexibility in the discretization of the domain, and independently, in the size of the numerical problem. The determination of multiple eigenvalues and eigenmodes is also straightforward. Finally, the so-called tension, that has to be minimized to find the eigenfrequencies, has a specific shape that can be used to speed up the search. Future improvements include sampling schemes yielding better stability of the numerical problems, and using this method to compute high frequency eigenmodes of plates in an efficient way.

Reproducible research

The Matlab/Octave code used to compute eigenfrequencies and eigenmodes of figures 3 and 4 is available online [20].

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References

1. J. T. Chen, S. Y. Lin, K. H. Chen, and I. L. Chen, “Mathematical analysis and numerical study of true and spurious eigenequations for free vibration of plates using real-part BEM,” *Computational Mechanics*, vol. 34, pp. 165–180, 2004.

2. C. J. S. Alves and P. R. S. Antunes, "The method of fundamental solutions applied to the calculation of eigensolutions for 2D plates," *International Journal for Numerical Methods in Engineering*, vol. 77, no. 2, pp. 177–194, 2009.
3. S. W. Kang and J. M. Lee, "Free vibration analysis of arbitrarily shaped plates with clamped edges using wave-type functions," *Journal of Sound and Vibration*, vol. 242, no. 1, pp. 9 – 26, 2001.
4. L. Fox, P. Henrici, and C. Moler, "Approximations and bounds for eigenvalues of elliptic operators," *SIAM Journal on Numerical Analysis*, vol. 4, no. 1, pp. 89–102, 1967.
5. T. Betcke and L. Trefethen, "Reviving the method of particular solutions," *SIAM review*, pp. 469–491, 2005.
6. A. H. Barnett, *Dissipation in Deforming Chaotic Billiards*. PhD thesis, Harvard University, 2000.
7. I. N. Vekua, *New methods for solving elliptic equations*. North-Holland, 1967.
8. P. Henrici, "A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients," *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, vol. 8, pp. 169–203, 1957. 10.1007/BF01600500.
9. A. Moiola, R. Hiptmair, and I. Perugia, "Vekua theory for the Helmholtz operator," *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, vol. 62, pp. 779–807, 2011.
10. J. Melenk, "Operator adapted spectral element methods I: harmonic and generalized harmonic polynomials," *Numerische Mathematik*, vol. 84, pp. 35–69, 1999.
11. A. Moiola, R. Hiptmair, and I. Perugia, "Plane wave approximation of homogeneous Helmholtz solutions," *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, vol. 62, pp. 809–837, 2011. 10.1007/s00033-011-0147-y.
12. A. Moiola, R. Hiptmair, and I. Perugia, "Approximation by plane waves," Research Report 2009-27, Eidgenössische Technische Hochschule Zürich, August 2009.
13. M. Géradin and D. Rixen, *Mechanical vibrations: theory and application to structural dynamics*. Wiley, 1997.
14. "Cast3M 12." http://www-cast3m.cea.fr/.
15. A. Leissa, *Vibration of Plates*. NASA, Washington, 1969.
16. J. Descloux and M. Tolley, "An accurate algorithm for computing the eigenvalues of a polygonal membrane," *Computer Methods in Applied Mechanics and Engineering*, vol. 39, pp. 37–53, 1983.
17. T. Betcke, *Numerical computation of eigenfunctions of planar regions*. PhD thesis, Keble College, University of Oxford, 2005.
18. C. Vanmaele, D. Vandepitte, and W. Desmet, "An efficient wave based prediction technique for dynamic plate bending problems with corner stress singularities," *Computer Methods in Applied Mechanics and Engineering*, vol. 198, pp. 2227–2245, 2009.
19. G. Chardon, A. Cohen, and L. Daudet, "Approximation of solutions to the Helmholtz equation from scattered data." Arxiv preprint 1301.0237.
20. http://gilleschardon.fr/mpscm.