Spatial correlations of near-field speckles generated by volume scattering

Rémi Carminati

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In this note we study the spatial correlation of the electric field close to the surface of a disordered medium. We consider monochromatic illumination (frequency ω) and a transmission geometry. The exit surface is *on average* a flat interface (when averaging over the statistical ensemble of realizations of the random medium). We arbitrarily choose this average surface as the plane $z = 0$. Our goal is the calculation of the spatial correlation function of two components of the electric field $\langle E_k(\mathbf{r})\, E_l^*(\mathbf{r}')\rangle$ in a plane at a given distance z. We focus on the mesoscopic regime, for which $\ell_\epsilon \ll \lambda \ll \ell \ll L$, where ℓ_ϵ is the correlation length of the medium (this quantity will be defined precisely below), λ is the wavelength, ℓ is the scattering mean-free path and L is the size of the system.

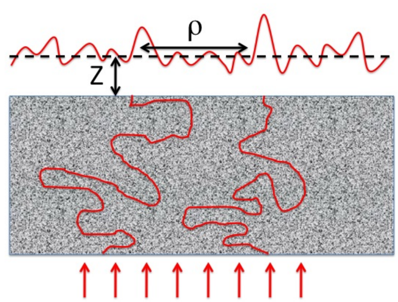

Figure 1: Schematic view of the geometry. The plane $z = 0$ is the averaged plane of the exit surface of the scattering medium. We compute the spatial correlation of the speckle in a plane at a distance z from the interface.

1 Electromagnetic wave scattering by a random medium

1.1 Model of random medium

We consider a disordered medium described by a dielectric function of the form $\epsilon(\mathbf{r}) = 1 + \delta\epsilon(\mathbf{r})$, where $\delta\epsilon(\mathbf{r})$ is the fluctuating part that satisfies $\langle \delta\epsilon(\mathbf{r})\rangle = 0$. The brackets $\langle ...\rangle$ denote a statistical averaging over an ensemble of realizations of the random medium. We assume a real dielectric function (no absorption).

In order to account for a certain level of correlation in the disordered medium, one introduces a correlation length ℓ_ϵ, through, e.g., a gaussian correlation function:

$$\begin{aligned} \langle \delta\epsilon(\mathbf{r}) \rangle &= 0 \\ \langle \delta\epsilon(\mathbf{r})\, \delta\epsilon(\mathbf{r}') \rangle &= \frac{U}{\pi^{3/2}\, \ell_\epsilon^3} \exp(-|\mathbf{r}-\mathbf{r}'|^2/\ell_\epsilon^2)\,. \end{aligned} \tag{1}$$

The constant U is determined by calculating the imaginary part of the effective dielectric function. The effective dielectric function is in general a dyadic function, and a non local quantity [1]:

$$\boldsymbol{\epsilon}_{\text{eff}}(\mathbf{k}) = \mathbf{I} + k_0^{-2}\boldsymbol{\Sigma}(\mathbf{k}) \tag{2}$$

where $k_0 = \omega/c = 2\pi/\lambda$ is the wavenumber in vacuum, and $\boldsymbol{\Sigma}(\mathbf{k})$ is the mass operator (or self energy). This quantity is the sum of all multiply connected scattering events, and can be expanded in a perturbative series in terms of $(k_0\ell)^{-1}$ [2]. Assuming $k_0\,\ell_\epsilon \ll 1$, the calculation of the effective dielectric function to lowest order in $(k_0\ell)^{-1}$ can be made explicitly (a similar calculation is performed in ref. [3] in the case of a disorder with an exponential correlation function). The dielectric function at this level of approximation is a scalar and local function, $\boldsymbol{\epsilon}_{\text{eff}}(\mathbf{k}) = \epsilon_{\text{eff}}\,\mathbf{I}$, where $\mathbf{I}$ is the unit tensor, and reads:

$$\epsilon_{\text{eff}} = 1 + \frac{k_0^2\, U}{3\pi^{3/2}\,\ell_\epsilon} + i\frac{U\, k_0^3}{6\pi}\,. \tag{3}$$

The imaginary term describes the attenuation of the amplitude of the averaged (or coherent) field by scattering. To first order in $(k_0\ell)^{-1}$, the imaginary part of the effective refractive index $n_{\text{eff}} = \sqrt{\epsilon_{\text{eff}}}$ is

$$\operatorname{Im} n_{\text{eff}} = \frac{k_0^3\, U}{12\pi} \equiv \frac{1}{2k_0\,\ell}\;. \tag{4}$$

The last equality is simply the definition of the scattering mean free path ℓ (in a non absorbing medium). This expression determines the constant U:

$$U = \frac{6\pi}{k_0^4\,\ell} \tag{5}$$

Finally, the effective dielectric function of the correlated medium reads:

$$\epsilon_{\text{eff}} = 1 + \frac{2}{\sqrt{\pi}\, k_0^2\,\ell\,\ell_\epsilon} + i\frac{1}{k_0\,\ell}\,. \tag{6}$$

We have already made use of the assumptions $k_0\,\ell \gg 1$ and $k_0\,\ell_\epsilon \ll 1$. This last expression shows that in order to satisfy the so-called weak scattering approximation, that is made in most of the studies of speckle correlations in random media [4, 5, 6, 7], one also needs $k_0^2\,\ell\,\ell_\epsilon \gg 1$ (to ensure that ϵ_{eff} does not deviate too much from unity).

Limiting case: White-noise gaussian statistics

In the limit $\ell_\epsilon \to 0$, one obtains the so-called white-noise gaussian statistics for the fluctuations of the dielectric constant:

$$\begin{aligned} \langle \delta\epsilon(\mathbf{r}) \rangle &= 0 \\ \langle \delta\epsilon(\mathbf{r})\, \delta\epsilon(\mathbf{r}') \rangle &= U\,\delta(\mathbf{r}-\mathbf{r}') \end{aligned} \tag{7}$$

This model is that used in most of the early calculations of speckle correlations [5, 6].

1.2 Basic equation for field fluctuations

The electric field in the medium obeys the vector propagation equation:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - \epsilon(\mathbf{r})\, k_0^2\, \mathbf{E}(\mathbf{r}) = 0\,. \tag{8}$$

We split the field into averaged and fluctuating parts:

$$\mathbf{E}(\mathbf{r}) = \langle \mathbf{E}(\mathbf{r}) \rangle + \delta\mathbf{E}(\mathbf{r}) \tag{9}$$

with $\langle \delta\mathbf{E}(\mathbf{r}) \rangle = 0$ by definition. The Dyson equation [1] states that in a statistically homogeneous medium, the averaged field obeys a propagation equation in the effective homogeneous medium described by the effective dielectric function ϵ_{eff} (to lowest order in $(k_0\,\ell)^{-1}$, ϵ_{eff} is a local and isotropic dielectric function):

$$\nabla \times \nabla \times \langle \mathbf{E}(\mathbf{r}) \rangle - \epsilon_{\text{eff}}\, k_0^2\, \langle \mathbf{E}(\mathbf{r}) \rangle = 0\,. \tag{10}$$

In order to find the propagation equation for the field fluctuations, we introduce (9) into (8), and make use of Eq. (10). We end up with

$$\nabla \times \nabla \times \delta\mathbf{E}(\mathbf{r}) - \epsilon_{\text{eff}}\, k_0^2\, \delta\mathbf{E}(\mathbf{r}) = (1 - \epsilon_{\text{eff}})\, k_0^2\, \mathbf{E}(\mathbf{r}) + \delta\epsilon(\mathbf{r})\, k_0^2\, \mathbf{E}(\mathbf{r})\,. \tag{11}$$

This equation has been obtained without any approximation. It shows that the field fluctuations $\delta\mathbf{E}$ obeys a propagation equation in the effective medium with a source term proportional to the total field $\mathbf{E}$ inside the medium.

1.3 Source term. Analogy with the Langevin model

In the right-hand side in Eq. (11), the first term is negligible compared to the second term. Indeed, $\epsilon_{\text{eff}} - 1 \sim 2/(\sqrt{\pi}\, k_0^2\, \ell\, \ell_\epsilon)$ and $\delta\epsilon \sim (k_0^4\, \ell\, \ell_\epsilon^3)^{-1/2} \sim (\epsilon_{\text{eff}} - 1)(\ell/\ell_\epsilon)^{1/2}$. Therefore, one has $\delta\epsilon \gg (\epsilon_{\text{eff}} - 1)$, recalling that $\ell_\epsilon \ll \lambda \ll \ell$. In the perturbative approach, Eq. (11) simplifies into:

$$\nabla \times \nabla \times \delta\mathbf{E}(\mathbf{r}) - \epsilon_{\text{eff}}\, k_0^2\, \delta\mathbf{E}(\mathbf{r}) = \delta\epsilon(\mathbf{r})\, k_0^2\, \mathbf{E}(\mathbf{r})\,. \tag{12}$$

The source term can be cast in the form of a current density:

$$\nabla \times \nabla \times \delta\mathbf{E}(\mathbf{r}) - \epsilon_{\text{eff}}\, k_0^2\, \delta\mathbf{E}(\mathbf{r}) = i\, \mu_0\, \omega\, \mathbf{j}(\mathbf{r}) \tag{13}$$

with $\mathbf{j}(\mathbf{r}) = -i\omega\, \epsilon_0\, \delta\epsilon(\mathbf{r})\, \mathbf{E}(\mathbf{r})$. The source term is a fluctuating quantity. In order to compute the electric field spatial correlation, we need to specify the spatial correlation function of the fluctuating currents:

$$\langle j_m(\mathbf{r})\, j_n^*(\mathbf{r}') \rangle = \epsilon_0^2\, \omega^2\, \langle \delta\epsilon(\mathbf{r})\, E_m(\mathbf{r})\, \delta\epsilon(\mathbf{r}')\, E_n^*(\mathbf{r}') \rangle\,. \tag{14}$$

The correlator in the right-hand side is a complex object. Using diagrammatic expansions, one shows that an operator exists, such that [1, 2]:

$$\langle \delta\epsilon(\mathbf{r})\, E_m(\mathbf{r})\, \delta\epsilon(\mathbf{r}')\, E_n^*(\mathbf{r}') \rangle = \int \Gamma_{mn,pq}(\mathbf{r}, \mathbf{r}', \mathbf{r}_1, \mathbf{r}_1')\, \langle E_p(\mathbf{r}_1)\, E_q^*(\mathbf{r}_1') \rangle\, \mathrm{d}^3\mathbf{r}_1\, \mathrm{d}^3\mathbf{r}_1'\,. \tag{15}$$

The operator $\Gamma(\mathbf{r}, \mathbf{r}', \mathbf{r}_1, \mathbf{r}_1')$ is the irreducible vertex (or intensity operator) that drives field correlations. It is a very complicated operator, that can only be evaluated approximatively. To lowest order in a perturbative expansion in terms of $(k_0\,\ell)^{-1}$, one has [1, 2, 4]:

$$\Gamma_{mn,pq}(\mathbf{r}, \mathbf{r}', \mathbf{r}_1, \mathbf{r}_1') \simeq \langle \delta\epsilon(\mathbf{r})\, \delta\epsilon(\mathbf{r}') \rangle\, \delta(\mathbf{r} - \mathbf{r}_1)\, \delta(\mathbf{r}' - \mathbf{r}_1')\, \delta_{mp}\, \delta_{nq}\,. \tag{16}$$

Equation (14) can be rewritten:

$$\langle j_k(\mathbf{r})\, j_l^*(\mathbf{r}')\rangle = \epsilon_0^2\,\omega^2\,\langle \delta\epsilon(\mathbf{r})\,\delta\epsilon(\mathbf{r}')\rangle\,\langle E_k(\mathbf{r})\,E_l^*(\mathbf{r}')\rangle\,\delta_{kl}\,. \tag{17}$$

The last factor δ_{kl} ensures the absence of polarization correlation in the source term (we assume that in the bulk, and for a medium that is statistically homogeneous and isotropic, no polarization correlation exist for the field). Finally, using the model of disorder introduced at the beginning of this note, we end up with the expression of the spatial correlation function of the current density, acting as a source term for the field fluctuations (valid under the conditions $k_0\,\ell \gg 1$, $k_0\,\ell_\epsilon \ll 1$, $k_0^2\,\ell\,\ell_\epsilon \gg 1$):

$$\langle j_k(\mathbf{r})\, j_l^*(\mathbf{r}')\rangle = \epsilon_0^2\,\omega^2\,\frac{6\pi}{\pi^{3/2}\,\ell_\epsilon^3\,k_0^4\,\ell}\,\exp(-|\mathbf{r}-\mathbf{r}'|^2/\ell_\epsilon^2)\,\langle E_k(\mathbf{r})\,E_l^*(\mathbf{r}')\rangle\,\delta_{kl} \tag{18}$$

In the limiting case $\ell_\epsilon \to 0$, we get the correlation function in the white-noise gaussian model:

$$\langle j_k(\mathbf{r})\, j_l^*(\mathbf{r}')\rangle = \epsilon_0^2\,\omega^2\,\frac{2\pi}{k_0^4\,\ell}\,\langle |\mathbf{E}|^2\rangle\,\delta(\mathbf{r}-\mathbf{r}')\,\delta_{kl} \tag{19}$$

where we have introduced $\langle |\mathbf{E}|^2\rangle = \sum_m \langle |E_m|^2\rangle$. Equations (13) and (18), or (13) and (19), are the basic equations for the calculation of the field spatial correlation. The model based on such a couple of equations is similar to the Langevin model initially developed to study Brownian motion [8]. The left hand-side in Eq. (13) applies to the field fluctuations, but is the same as that describing the averaged field (Eq .10). The source term in Eq. (13) is a fluctuating source, similar to the Langevin forces. In conclusion, we have constructed a Langevin model that will allow us to compute the spatial correlation of the electric field in various situations. The construction of this model has required some approximations, that have been clearly specified. Finally, we stress that Langevin models have already been used to describe many fluctuational phenomena in wave scattering, including thermal electromagnetic fields fluctuations [9, 10, 11] and long-range correlations in speckle patterns using the diffusion approximation [1, 6].

1.4 Calculation of the field spatial correlation - Green function

The averaged Green function $\langle \mathbf{G}\rangle$, solution of the Dyson equation [1] with an outgoing wave condition at infinity, satisfies:

$$\nabla\times\nabla\times\langle \mathbf{G}(\mathbf{r},\mathbf{r}')\rangle - \epsilon_{\text{eff}}\,k_0^2\,\langle \mathbf{G}(\mathbf{r},\mathbf{r}')\rangle = \delta(\mathbf{r}-\mathbf{r}')\,. \tag{20}$$

The solution of Eq. (13) can be written using the averaged dyadic Green function:

$$\delta\mathbf{E}(\mathbf{r}) = i\mu_0\,\omega\int \langle \mathbf{G}(\mathbf{r},\mathbf{r}_1)\rangle\,\mathbf{j}(\mathbf{r}_1)\,\mathrm{d}^3 r_1 \tag{21}$$

where the integral is extended to the volume of the scattering medium. For a medium with large optical thickness, we can neglect the exponentially small contribution of the averaged field, so that $\langle \delta E_k(\mathbf{r})\,\delta E_l^*(\mathbf{r}')\rangle = \langle E_k(\mathbf{r})\,E_l^*(\mathbf{r}')\rangle$. The dyadic Green function is known analytically in the case of an infinite homogeneous medium, and in the case of a flat interface separating two homogeneous media. Both situations are of interest for the present study. The field spatial correlation reads (we assume a summation over repeated indices)

$$\boxed{\langle E_k(\mathbf{r})\,E_l^*(\mathbf{r}')\rangle = \mu_0^2\,\omega^2\int_V \langle G_{km}(\mathbf{r},\mathbf{r}_1)\rangle\,\langle G_{ln}^*(\mathbf{r}',\mathbf{r}_1')\rangle\,\langle j_m(\mathbf{r}_1)\,j_n^*(\mathbf{r}_1')\rangle\,\mathrm{d}^3 r_1\mathrm{d}^3 r_1'\,.} \tag{22}$$

This equation is the starting point of all the calculations that follow. We will first study the case of an infinite homogeneous medium, and recover the result already derived by Shapiro [5] (extended to the case of vector fields). Then we will study the case of a flat interface, in order to study the spatial correlations in the near field of the random medium.

2 Spatial correlation in an infinite medium

We first perform the calculation in the case of the white-noise gaussian statistics (i.e. at the same level of approximation as the first calculation by Shapiro [5]). Then we show that the same result is obtained in the case of a medium with a finite correlation length.

2.1 White-noise gaussian statistics

In this case, inserting Eq. (19) into Eq. (22) yields:

$$\langle E_k(\mathbf{r})\, E_l^*(\mathbf{r}')\rangle = \frac{2\pi}{\ell}\,\langle|\mathbf{E}|^2\rangle \int_V \langle G_{km}(\mathbf{r},\mathbf{r}_1)\rangle\,\langle G_{lm}^*(\mathbf{r}',\mathbf{r}_1)\rangle\,\mathrm{d}^3r_1\,. \tag{23}$$

Using the identity [12]

$$k_0^2\int_V \mathrm{Im}\epsilon_{\mathrm{eff}}\,\langle G_{km}(\mathbf{r},\mathbf{r}_1)\rangle\,\langle G_{lm}^*(\mathbf{r}',\mathbf{r}_1)\rangle\,\mathrm{d}^3r_1 = \mathrm{Im}\langle G_{kl}(\mathbf{r},\mathbf{r}')\rangle \tag{24}$$

we obtain

$$\langle E_k(\mathbf{r})\, E_l^*(\mathbf{r}')\rangle = \frac{2\pi}{k_0}\,\langle|\mathbf{E}|^2\rangle\,\mathrm{Im}\langle G_{kl}(\mathbf{r},\mathbf{r}')\rangle\ . \tag{25}$$

This expression connects the field spatial correlation to the imaginary part of the Green function. This is a very general result in statistically homogeneous media, that has been used in acoustics for imaging applications [13]. In order to introduce a quantity that measures the spatial correlation of a vector field (that would be similar to that used for scalar fields), we define the degree of spatial coherence by

$$\gamma_E(\mathbf{r},\mathbf{r}') \equiv \sum_k \langle E_k(\mathbf{r})\, E_k^*(\mathbf{r}')\rangle = \frac{2\pi}{k_0}\,\langle|\mathbf{E}|^2\rangle\,\mathrm{Im}\,Tr\,\langle\mathbf{G}(\mathbf{r},\mathbf{r}')\rangle \tag{26}$$

where Tr denotes the trace of a tensor. For $\mathbf{r}=\mathbf{r}'$, we have $\gamma_E(\mathbf{r},\mathbf{r}) = \langle|\mathbf{E}|^2\rangle$. In an infinite medium that is statistically homogeneous and isotropic, the averaged Green function has an analytical expression [14, 15]:

$$\langle\mathbf{G}(\mathbf{r}-\mathbf{r}')\rangle = \left[\mathbf{I}+\frac{1}{k_{\mathrm{eff}}^2}\nabla\nabla\right]\frac{\exp(ik_{\mathrm{eff}}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{27}$$

for $\mathbf{r}\neq\mathbf{r}'$. From Eqs. (25-27), we obtain

$$\gamma_E(\rho) = \langle|\mathbf{E}|^2\rangle\,\mathrm{sinc}\left(\frac{2\pi}{\lambda_{\mathrm{eff}}}\rho\right)\,\exp\left(-\frac{\rho}{2\ell}\right) \tag{28}$$

where $\rho = |\mathbf{r}-\mathbf{r}'|$ and λ_{eff} is the effective wavelength in the medium, such that $2\pi/\lambda_{\mathrm{eff}} = k_0\,\mathrm{Re}\,n_{\mathrm{eff}}$. This expression is identical to that obtained in ref. [5] using diagrammatic techniques. The same expression is obtained for blackbody radiation in a weakly absorbing homogeneous medium [10].

2.2 Correlated medium

It is often stated that the correlation function (28) is universal, in the sense that it does not depend on the microscopic structure of the random medium. We shall now show that the same result is recovered in the case of a correlated random medium. Inserting Eq. (18) into Eq. (22), we obtain:

$$\langle E_k(\mathbf{r})\, E_l^*(\mathbf{r}')\rangle = \mu_0^2\,\omega^2 \int_V \langle G_{km}(\mathbf{r},\mathbf{r}_1)\rangle\, \langle G_{ln}^*(\mathbf{r}',\mathbf{r}_1')\rangle\, C_{mn}(|\mathbf{r}_1-\mathbf{r}_1'|)\,\delta_{mn}\,\mathrm{d}^3r_1\mathrm{d}^3r_1' \qquad (29)$$

where we have written the correlation of the currents in the form $\langle j_m(\mathbf{r}_1)\, j_n^*(\mathbf{r}_1')\rangle = C_{mn}(|\mathbf{r}_1 - \mathbf{r}_1'|)\,\delta_{mn}$. We now perform the following change of variable, with Jacobian unity: $\mathbf{R} = (\mathbf{r}_1+\mathbf{r}_1')/2$, $\mathbf{X} = \mathbf{r}_1 - \mathbf{r}_1'$:

$$\begin{aligned}
\langle E_k(\mathbf{r})\, E_l^*(\mathbf{r}')\rangle &= \mu_0^2\,\omega^2 \int_V \langle G_{km}(\mathbf{r},\mathbf{R}+\mathbf{X}/2)\rangle\, \langle G_{lm}^*(\mathbf{r}',\mathbf{R}-\mathbf{X}/2)\rangle\, C_{mm}(|\mathbf{X}|)\,\mathrm{d}^3R\,\mathrm{d}^3X \\
&= \mu_0^2\,\omega^2 \int \mathrm{d}^3X\, C_{mm}(|\mathbf{X}|) \int \langle G_{km}(\mathbf{r}-\mathbf{R}-\mathbf{X}/2)\rangle\, \langle G_{lm}^*(\mathbf{r}'-\mathbf{R}+\mathbf{X}/2)\rangle\,\mathrm{d}^3R \\
&= \frac{\mu_0^2\,\omega^2}{k_0^2\,\mathrm{Im}\,\epsilon_{\mathrm{eff}}} \int \mathrm{Im}\,\langle G_{kl}(\mathbf{r}-\mathbf{r}'-\mathbf{X})\rangle\, C_{mm}(|\mathbf{X}|)\,\mathrm{d}^3X\,. \qquad (30)
\end{aligned}$$

In the last equality we have used relation (24). The result takes the form of a convolution product. We know from the preceding calculation that $\mathrm{Im}\,\langle G_{kl}\rangle$ varies on the scales of λ and ℓ. The correlation of the currents is of the form:

$$C_{mn}(|\mathbf{X}|) = \epsilon_0^2\,\omega^2\,\frac{6\pi}{\pi^{3/2}\,k_0^4\,\ell\,\ell_\epsilon^3}\,\exp(-|\mathbf{X}|^2/\ell_\epsilon^2)\,\langle E_m(0)\,E_n^*(\mathbf{X})\rangle \qquad (31)$$

and varies on the scale of ℓ_ϵ. The condition $\ell_\epsilon \ll \lambda \ll \ell$ allows to simplify the convolution product:

$$\langle E_k(\mathbf{r})\, E_l^*(\mathbf{r}')\rangle \simeq \frac{6\pi}{k_0}\mathrm{Im}\,\langle G_{kl}(\mathbf{r}-\mathbf{r}')\rangle \int \frac{\exp(-|\mathbf{X}|^2/\ell_\epsilon^2)}{\pi^{3/2}\,\ell_\epsilon^3}\,\langle E_m(0)\,E_m^*(\mathbf{X})\rangle\,\mathrm{d}^3X\,. \qquad (32)$$

The gaussian function acts as a Dirac delta function with respect to the field correlation. We end up with:

$$\langle E_k(\mathbf{r})\, E_l^*(\mathbf{r}')\rangle = \frac{2\pi}{k_0}\mathrm{Im}\,\langle G_{kl}(\mathbf{r}-\mathbf{r}')\rangle\,\langle|\mathbf{E}|^2\rangle\,. \qquad (33)$$

We recover Eq. (25), and the subsequent results obtained in the case of the white-noise gaussian model. This calculation confirms that the expression (28) of the degree of spatial coherence in an infinite medium is independent on the correlation length of the medium. It is given by:

$$\boxed{\gamma_E(\rho) = \langle|\mathbf{E}|^2\rangle\,\mathrm{sinc}\left(\frac{2\pi}{\lambda_{\mathrm{eff}}}\rho\right)\,\exp\left(-\frac{\rho}{2\ell}\right)}\ \text{(infinite medium)} \qquad (34)$$

The universality of this form of the correlation function is derived in ref. [16].

3 Near-field spatial correlations (uncorrelated medium)

We now turn to the study of the field spatial correlation in the near field of the surface of a random medium. To do this, we only need to change the averaged Green function. Instead of using the Green function of an infinite homogenous medium, we use the Green function for an interface separating the effective random medium (medium $z < 0$ described by a homogeneous dielectric constant ϵ_{eff}) from a vacuum (medium $z > 0$)[1]. This Green function connects the current density for $z < 0$ to the field in vacuum for $z > 0$. Its expression [17] involves the Fresnel transmission factors at the interface, and is given in Appendix A.

We first study the case of an uncorrelated medium (white-noise gaussian model). This means that we focus on distances z such that $\ell_\epsilon \ll z$. We study both the far-field regime $\ell_\epsilon \ll \lambda \ll z$ and the near-field regime $\ell_\epsilon \ll z \ll \lambda$ (this regime only exists if ℓ_ϵ is very small compared to λ).

3.1 General expression

We insert Eqs. (47) and (48) of Appendix A into Eq. (23). We calculate the degree of spatial coherence of the field $\gamma_E(\mathbf{r}, \mathbf{r}')$ in a plane parallel to the interface, at a distance z. After some tedious algebra, we get

$$\gamma_E(\mathbf{r}, \mathbf{r}') = \frac{\langle |\mathbf{E}|^2 \rangle}{4\ell} \int_0^\infty f(K, z)\, J_0(K\rho)\, \mathrm{d}K \tag{35}$$

where the function in the integrand is

$$f(K, z) = \frac{K}{2\mathrm{Im}\, \gamma_2} \left[\frac{|t_s|^2}{|\gamma_2|^2} + \frac{(K^2 + |\gamma_1|^2)(K^2 + |\gamma_2|^2)}{|n_{\text{eff}}|^2\, k_0^4} \frac{|t_p|^2}{|\gamma_2|^2} \right] \exp(-2\mathrm{Im}\, \gamma_1\, z)\,. \tag{36}$$

In this expression, $\gamma_1 = (k_0^2 - K^2)^{1/2}$ is the z-component of the wavevector in vacuum, and $\gamma_2 = (\epsilon_{\text{eff}}\, k_0^2 - K^2)^{1/2}$ is the z-component of the wavevector in the effective medium. The square-roots are chosen so that $\mathrm{Im}\, \gamma_1 > 0$ and $\mathrm{Im}\, \gamma_2 > 0$. Equation (35) is exact, and can be used to compute the spatial correlation at any distance z from the interface.

3.2 Far-field behavior

In the far field ($z \gg \lambda$), the exponential filter $\exp(-2\mathrm{Im}\, \gamma_1\, z)$ reduces the range of $f(K, z)$ in Eq. (35) to $0 \leq K \leq k_0$, with $k_0 = \omega/c = 2\pi/\lambda$ (propagating waves). The expression of the degree of spatial coherence can be simplified (useful formulas that allow us to simplify the expression of $f(K, z)$ are taken from [18]). One obtains:

$$\gamma_E(\mathbf{r}, \mathbf{r}') = \frac{\langle |\mathbf{E}|^2 \rangle}{4k_0} \int_0^{k_0} \frac{K}{\sqrt{k_0^2 - K^2}} \left[2 - r_s(K) - r_p(K)\right] J_0(K\rho)\, \mathrm{d}K \tag{37}$$

where r_s and r_p are the Fresnel reflection factors of the effective medium at the interface $z = 0$, for s and p polarization (given in Appendix A). This expression is exact in the far field (we have only assumed that $K \leq k_0$ in Eq. 35). If we define an intensity reflection factor averaged over polarizations $R = [|r_s(K)|^2 + |r_p(K)|^2]/2$, we can reasonably assume that R as a weak dependence on K (the effective medium is such that $n_{\text{eff}} \simeq 1$ so that the angular dependence of the reflection factor is weak). In this case, the integral simplifies:

$$\gamma_E(\mathbf{r}, \mathbf{r}') = \frac{\langle |\mathbf{E}|^2 \rangle}{2k_0} (1 - R) \int_0^{k_0} \frac{K}{\sqrt{k_0^2 - K^2}}\, J_0(K\rho)\, \mathrm{d}K \tag{38}$$

[1] We recall that the plane $z = 0$ is defined as the *averaged* interface between the random medium and vacuum.

The integral can be calculated analytically, and equals $k_0 \, \mathrm{sinc}(k_0\rho)$ where $\mathrm{sinc}(x) = \sin(x)/x$. We end-up with the far-field expression of the field correlation function:

$$\boxed{\gamma_E(\mathbf{r}, \mathbf{r}') = \frac{(1-R)\,\langle|\mathbf{E}|^2\rangle}{2} \, \mathrm{sinc}(k_0\rho)} \text{ for } z \gg \lambda \text{ (far field)} \tag{39}$$

where $\rho = |\mathbf{r} - \mathbf{r}'|$ in a plane at a constant height z. In the far field, the correlation function is independent on z. The $\mathrm{sinc}(k_0\rho)$ behavior is similar to that obtained for blackbody radiation in the far field of a planar thermal source [11]. The width of the correlation function is $\lambda/2$. Also note that the correlation function does not exhibit the $\exp(-\rho/2l)$ term obtained in an infinite medium.

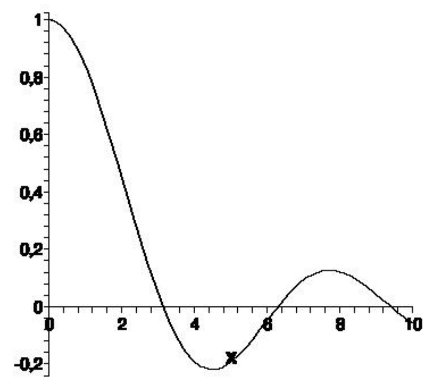

Figure 2: Plot of the function $\mathrm{sinc}(x)$.

3.3 Near-field regime: quasi-static limit

We now study the short-distance regime. When $z \ll \lambda$, inspecting the integrand in Eq. (35), we see that large wavevectors satisfying $1/z \gg K \gg k_0$ dominate the integral. This is because the exponential cut-off for large K is $\exp(-2\mathrm{Im}\,\gamma_1\, z) \simeq \exp(-2Kz)$, and becomes effective only for $K \gg 1/z$. The asymptotic behavior of the integral can therefore be determined by expanding the integrand to leading order in the limit $K \gg k_0$, known as the quasi-static limit. In this limit, one has $\gamma_1 = \gamma_2 \simeq i|K|$, allowing us to simplify the expression of $f(K, z)$. We finally obtain

$$\gamma_E(\mathbf{r}, \mathbf{r}') \approx \frac{2\,\langle|\mathbf{E}|^2\rangle}{k_0^4\,\ell\,|\epsilon_{\mathrm{eff}}+1|^2} \int_0^\infty K^2 \, \exp(-2Kz)\, J_0(K\rho)\,\mathrm{d}K \; . \tag{40}$$

The integral can be expressed from Eq. (51) in Appendix B, taking the second derivative with respect to z. Moreover we can make the approximation $|\epsilon_{\mathrm{eff}}+1|^2 \simeq 4$ in the denominator. We end up with

$$\boxed{\gamma_E(\mathbf{r}, \mathbf{r}') \approx \frac{\langle|\mathbf{E}|^2\rangle}{8k_0^4\,\ell\,z^3} \; \frac{1-\rho^2/(8z^2)}{[1+\rho^2/(4z^2)]^{5/2}}} \text{ for } \ell_\epsilon \ll z \ll \lambda \text{ (near field)} \tag{41}$$

where we have used the notation $\rho = |\mathbf{r}-\mathbf{r}'|$, with $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z)$. Remember that $\langle|\mathbf{E}|^2\rangle$ in this equation is the average intensity in the infinite medium. The last term describes the spatial dependence, and has a width on the order of z (see Fig. 3). In the near field, and assuming an uncorrelated medium (i.e. a correlation length of the medium much smaller than the distance z), we have a spatial correlation length of the field on the order of z. Also note that this quasi-static contribution to the correlation length increases as $1/z^3$. The same behavior has

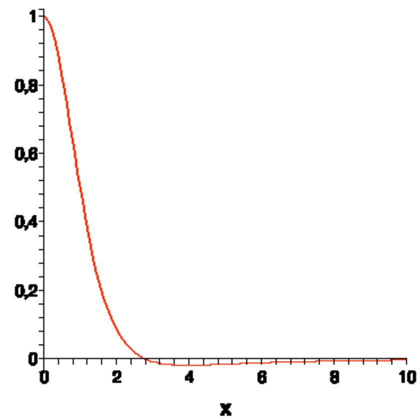

Figure 3: Plot of the normalized correlation function in the quasi-static approximation (uncorrelated medium), calculated at a distance z. The variable x is ρ/z, with $\rho = |\mathbf{r} - \mathbf{r}'|$.

been obtained for thermal near fields [19]. As explained in [19], it can be understood from the quasi-static behavior of the field at distances much smaller than the wavelength. In this regime, the field has a spatial dependence that is the same as the electrostatic field. An electrostatic point source at a distance z from a plane generates a field on a zone of typical lateral extension z (due to the $1/z^3$ decay of the field). By reciprocity, a point detector placed at a distance z from the surface sees a source area on the surface on the order of πz^2. Another point detector at the same distance sees a source region of the same size. Correlation between the fields detected by the two detectors appears when the two source regions overlap, i.e., for a separation smaller than z.

4 Influence of the correlation length of the medium

The behavior described in the preceding section could be seen only in a in a situation in which $\ell_\epsilon \ll z \ll \lambda$, where ℓ_ϵ is the (geometrical) correlation length of the disordered medium. We now examine the behavior of the field correlation function in the regime $z \sim \ell_\epsilon$, and even $z < \ell_\epsilon$.

4.1 General expression

For a finite correlation length ℓ_ϵ, the spatial correlation of the currents is given by Eq. (18). We have shown that in an infinite medium (or equivalently in the bulk of the semi-infinite medium), the field spatial correlation varies of the scale of λ or ℓ, both being much larger than

ℓ_ϵ. Therefore, the correlation function of the currents can be simplified into:

$$\langle j_k(\mathbf{r})\, j_l^*(\mathbf{r}')\rangle = \epsilon_0^2\,\omega^2\,\frac{2\pi}{\pi^{3/2}\,\ell_\epsilon^3\,k_0^4\,\ell}\,\exp(-|\mathbf{r}-\mathbf{r}'|^2/\ell_\epsilon^2)\,\langle|\mathbf{E}|^2\rangle\,\delta_{kl}\,. \tag{42}$$

Inserting Eq. (42) and the expression of the Green function Eqs. (47-48) into Eq. (22), we obtain, after some algebra

$$\begin{aligned}\langle E_k(\mathbf{r})\,E_l^*(\mathbf{r}')\rangle &= \pi\frac{\langle|\mathbf{E}|^2\rangle}{\ell}\int\frac{1}{\mathrm{Im}\,\gamma_2}g_{km}(\mathbf{K})\,g_{lm}^*(\mathbf{K})\,\exp[i\mathbf{K}\cdot(\mathbf{R}-\mathbf{R}')]\\ &\times\ \exp(-2\mathrm{Im}\,\gamma_1\,z)\,\exp(-K^2\,\ell_\epsilon^2/4)\,\exp[-(\mathrm{Re}\,\gamma_2)^2\,\ell_\epsilon^2/4]\,\mathrm{d}^2K\,. \end{aligned} \tag{43}$$

As in the previous section, we calculate the degree of spatial coherence of the field. Using the fact that the integrand only depends on $K=|\mathbf{K}|$, we can simplify the expression that becomes [2]

$$\gamma_E(\mathbf{r},\mathbf{r}') = \frac{\langle|\mathbf{E}|^2\rangle}{4\ell}\int_0^\infty f(K,z)\,\exp[-(K^2+(\mathrm{Re}\,\gamma_2)^2)\,\ell_\epsilon^2/4]\,J_0(K\rho)\,\mathrm{d}K \tag{44}$$

Note that for $\ell_\epsilon=0$ (uncorrelated medium), we recover expression (35). The main difference with the previous situation is the existence of an exponential cut-off for $K>1/\ell_\epsilon$.

4.2 Short-distance limit

We now examine the regime $z\sim\ell_\epsilon$. The quasi-static approximation is valid (we recall that $\ell_\epsilon\ll\lambda$), and the same approximations as that leading to Eq. (40) yield

$$\gamma_E(\mathbf{r},\mathbf{r}') \approx \frac{2\,\langle|\mathbf{E}|^2\rangle}{k_0^4\,\ell\,|\epsilon_{\mathrm{eff}}+1|^2}\int_0^\infty K^2\,\exp(-2Kz)\,\exp(-K^2\,\ell_\epsilon^2/4)\,J_0(K\rho)\,\mathrm{d}K\ . \tag{45}$$

For $\ell_\epsilon=0$, we have seen that the integral gives a function with a lateral extent on the order of z, and an amplitude that increases as $1/z^3$ (Eq. 41). Due to the exponential cut-off $\exp(-K^2\,\ell_\epsilon^2/4)$, when $z\sim\ell_\epsilon$ we expect a crossover to a regime that becomes independent on z and driven by the length scale ℓ_ϵ. Indeed, when $z<\ell_\epsilon$, the exponential term $\exp(-2Kz)$ is close to unity. In this case, the integral in Eq. (45) is actually a hypergeometric confluent function $M(a,b,x)$ (see the formula in Appendix B). We finally get

$$\boxed{\gamma_E(\mathbf{r},\mathbf{r}') \approx \frac{\sqrt{\pi}\,\langle|\mathbf{E}|^2\rangle}{k_0^4\,\ell\,\ell_\epsilon^3}\ M\left(\frac{3}{2},1,\frac{-\rho^2}{\ell_\epsilon^2}\right)}\ \text{for } z\lesssim\ell_\epsilon\ll\lambda \text{ (near field)} \tag{46}$$

where we have made the approximation $|\epsilon_{\mathrm{eff}}+1|^2\simeq 4$. The last term describes the spatial dependence, and has a width on the order of ℓ_ϵ (see Fig. 4). For a medium with a geometric correlation length ℓ_ϵ much smaller than λ, the correlation length of the near field is on the order of ℓ_ϵ when $z\lesssim\ell_\epsilon$. Also note that the correlation function does not depend on z any more. The spatial-frequency cut-off due to ℓ_ϵ has removed the $1/z^3$ increase of the quasi-static contribution calculated with $\ell_\epsilon=0$.

[2] The use of a local effective dielectric in this regime might be questioned. It is known that on scales on the order of ℓ_ϵ, ϵ_{eff} becomes a non-local function [1]. In fact, the exponential term $\exp(-K^2\,\ell_\epsilon^2/4)$ in Eq. (43) prevents high spatial frequencies $K>1/\ell_\epsilon$ from playing a role so that the effect of non-locality should be small. Our model should give a good estimate.

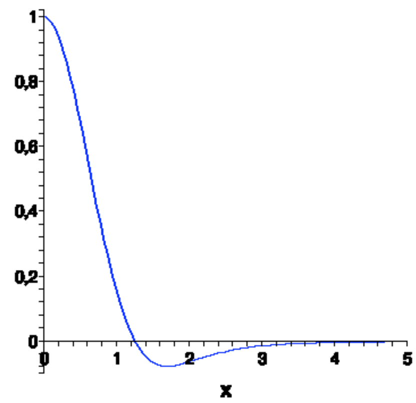

Figure 4: Plot of the normalized correlation function in the quasi-static approximation for a medium with a correlation length ℓ_ϵ. The variable x is ρ/ℓ_ϵ, with $\rho = |\mathbf{r} - \mathbf{r}'|$.

Summary

If we assume to have a random medium with a (geometric) spatial correlation that can be described by a correlation length ℓ_ϵ, with $\ell_\epsilon \ll \lambda \ll \ell$, our model predicts three regimes for the field spatial correlation in a plane at a distance z from the averaged surface of the medium:

1. For $z \gg \lambda$ (far field), the spatial correlation is on the order of $\lambda/2$, with λ the wavelength in vacuum. The degree of spatial coherence behaves as $\text{sinc}(2\pi\rho/\lambda)$, where ρ is the distance between the two observation points (see Eq. 39).

2. For $\ell_\epsilon \ll z \ll \lambda$, a near-field regime exists in which neither the correlation length of the disorder nor the wavelength play a role. The quasi-static fields generated by the effective (homogeneous) medium create a spatial correlation of the field on a length scale on the order of z (see Eq. 41).

3. When $z \sim \ell_\epsilon$, one observes a crossover between the previous regime and a near-field regime that is driven by the length scale ℓ_ϵ. For $z < \ell_\epsilon$, the spatial correlation of the field does not depend on z any more. The field is correlated on a length scale on the order of the correlation length of the disordered medium (see Eq. 46).

Comments

- In the very near field, the spatial correlation function of the field is strongly connected to the spatial correlation of the random medium. This behavior has been discussed previously in the case of near-fields generated by random rough surfaces [21, 22].

- In practice, regime (2) could be seen only if the correlation length of the medium ℓ_ϵ is really very small compared to λ. If this is not the case, the transition from the far-field regime (regime (1)) to the regime driven by ℓ_ϵ (regime (3)) might occur on a range of distances z that is too small to allow a measurement of regime (2).

- When dealing with speckle correlations, one usually consider the intensity correlation $\langle I(\mathbf{r})I(\mathbf{r}')\rangle$. The width of this function is a measure of the averaged speckle spot size. In the present approach, we have calculated *field* correlations, as in the study of spatial coherence. In general, this is not the same. In the case of the short-range correlation giving the speckle spot size, one has $\langle I(\mathbf{r})I(\mathbf{r}')\rangle \propto |\gamma_E(\mathbf{r},\mathbf{r}')|^2$, so that the intensity correlation function can be deduced from the field correlation. This is sometimes called the "factorization" approximation [6]. This property can be deduced if the field is a gaussian random variable (in this case high order correlations can be written as products of second order correlations).

- There are a few experimental studies of near-field speckle properties based on SNOM measurements on volume disordered samples [23, 24] and on semicontinuous metal films supporting resonant excitations [25, 26]. In the case of disordered metal films, localized spots of nanometer size have been observed [27], attributed to Anderson localization (a regime that is not covered by the approach reported in this note). In the case of dielectric volume-disordered systems, subwalength speckle spots seems to have been observed [23], but on samples whose scattering properties have not been characterized.

Appendices

A Expression of the dyadic Green function for a plane interface

In this Appendix, we give the expression of the dyadic Green function for a plane interface separating a vacuum (medium $z > 0$, denoted by medium 1) from a semi-infinite medium with dielectric constant ϵ ((medium $z < 0$, denoted by medium 2).

The expression of the Green dyadic connecting a source at $\mathbf{r}'$ inside the medium to the field at $\mathbf{r}$ in the upper medium is [17]:

$$\mathbf{G}(\mathbf{r},\mathbf{r}',\omega) = \int \mathbf{g}(\mathbf{K},z,z') \exp[i\mathbf{K}\cdot(\mathbf{R}-\mathbf{R}')]\,\mathrm{d}^2K \tag{47}$$

where the Fourier transform $\mathbf{g}(\mathbf{K},z,z')$ reads

$$\mathbf{g}(\mathbf{K},z,z') = \frac{i}{8\pi^2}\frac{1}{\gamma_2}(\hat{\mathbf{s}}t_s\hat{\mathbf{s}} + \hat{\mathbf{p}}_1 t_p \hat{\mathbf{p}}_2)\exp(i\gamma_1 z - i\gamma_2 z')\,. \tag{48}$$

In this expression, $\mathbf{r} = (\mathbf{R}, z)$, $\hat{\mathbf{s}} = \hat{\mathbf{K}} \times \hat{z}$, $\hat{\mathbf{p}}_j = (|\mathbf{K}|\hat{z} + \gamma_j \hat{\mathbf{K}})/k_j$, the symbol $\hat{}$ denoting a unit vector, $k_1 = k = \omega/c$, $k_2 = \sqrt{\epsilon}\,k$, $\gamma_j = (k_j^2 - \mathbf{K}^2)^{1/2}$, with the determination $\mathrm{Re}(\gamma_j) > 0$ and $\mathrm{Im}(\gamma_j) > 0$. $\hat{\mathbf{s}}\hat{\mathbf{s}}$ is the dyadic defined by $(\hat{\mathbf{s}}\hat{\mathbf{s}})\cdot\mathbf{u} = (\hat{\mathbf{s}}\cdot\mathbf{u})\hat{\mathbf{s}}$.

The Fresnel transmission factors $t_s(\mathbf{K})$ and $t_p(\mathbf{K})$, for s and p polarization respectively, are given by

$$t_s(\mathbf{K}) = \frac{2\gamma_2}{\gamma_1+\gamma_2} \qquad t_p(\mathbf{K}) = \frac{2\sqrt{\epsilon}\gamma_2}{\epsilon\gamma_1+\gamma_2} \tag{49}$$

The Fresnel reflection factors $r_s(\mathbf{K})$ and $r_p(\mathbf{K})$, that are also used in some calculations, are given by

$$r_s(\mathbf{K}) = \frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2} \qquad r_p(\mathbf{K}) = \frac{\epsilon\gamma_1-\gamma_2}{\epsilon\gamma_1+\gamma_2} \tag{50}$$

B Useful formulas

$$\int_0^\infty J_0(K\rho)\exp(-2Kz)\,\mathrm{d}K = \frac{1}{(4z^2+\rho^2)^{1/2}} \tag{51}$$

$$\int_0^\infty K^2\exp(-K^2\,\delta^2/4)J_0(K\rho)\,\mathrm{d}K = \frac{2\sqrt{\pi}}{\delta^3}\,M\left(\frac{3}{2},1,\frac{-\rho^2}{\delta^2}\right) \tag{52}$$

$$\int_{-\infty}^{+\infty}\exp(-x^2/a^2)\exp(ikx)\,\mathrm{d}x = a\sqrt{\pi}\exp[-(k^2\,\delta^2)/4] \tag{53}$$

$$\delta(x) = \lim_{\epsilon\to 0^+}\frac{1}{2\sqrt{\pi\epsilon}}\exp[-x^2/(4\epsilon)] \tag{54}$$

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