Analysis of coherence properties of partially polarized light in 3D and application to disordered media

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The theory of the intrinsic coherence, originally developed for 2D fields, is generalized in order to analyze coherence properties of light with a polarization that can fluctuate in three dimensions. Several notions, such as the concept of mean-square coherence and the capacity to describe irreversible behaviors, are demonstrated and illustrated with the example of light in 3D disordered media with frozen and nonfrozen disorders.

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Coherence properties [1,2] of partially polarized light have been the subject of several investigations in the last decade [3–8]. Motivated initially by the analysis of beam propagation in free space or in atmospheric turbulence [9,10], these studies have generally been restricted to paraxial electromagnetic fields, where the field polarization can fluctuate only in 2D. However, there exist several physical situations in which the polarization can fluctuate in 3D. This can occur, for instance, in the near field of planar structures [11–13], or in multiple-scattering (disordered) media [14–16]. Such situations are commonly encountered and have already prompted investigations on, e.g., the definition of a 3D degree of polarization [17–20].

This Letter proposes a generalization of the intrinsic coherence theory to 3D fluctuating fields. Several coherence properties of partially polarized light, already known in 2D, are demonstrated in 3D. This is the case, in particular, for the mean-square coherence [21,22], and for the analysis of the irreversible behavior of light when it is subjected to random linear transformations [23,24] described with matrix vector multiplications. These properties are illustrated with the example of light propagation in disordered media with frozen and nonfrozen disorder, where an emphasis is given to the proper application of disorder averages in the analysis of intrinsic coherence.

The field in the time or in the frequency domain and at two locations, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, will be written $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$. The field is assumed statistically circular [1] and to fluctuate in 3D. Thus

$$\mathbf{E}(\boldsymbol{r}) = (E_x(\boldsymbol{r}), E_y(\boldsymbol{r}), E_z(\boldsymbol{r}))^T, \tag{1}$$

where $\boldsymbol{a}^T$ denotes the transpose vector of $\boldsymbol{a}$. The mutual coherence and the polarization matrices are, respectively,

$$\Omega(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle \mathbf{E}(\boldsymbol{r}_1)\mathbf{E}(\boldsymbol{r}_2)^\dagger \rangle \tag{2}$$

and

$$\mathbf{\Gamma}(\boldsymbol{r}_i) = \langle \mathbf{E}(\boldsymbol{r}_i)\mathbf{E}(\boldsymbol{r}_i)^\dagger \rangle, \tag{3}$$

where $\boldsymbol{a}^\dagger$ is the conjugate transpose of $\boldsymbol{a}$ and where $\langle\rangle$ is the statistical average. In the spectral domain, $\Omega(\boldsymbol{r}_1, \boldsymbol{r}_2)$ and $\mathbf{\Gamma}(\boldsymbol{r}_i)$ correspond, respectively, to the spectral coherence and to the spectral polarization matrices.

It can be interesting to analyze coherence properties of partially polarized fields with measures of coherence. The intrinsic degrees of coherence possess several interesting properties that can be generalized to fields in 3D. Mathematically, the intrinsic degrees of coherence $\mu_{E,j}(\boldsymbol{r}_1, \boldsymbol{r}_2)$, with $j = 1, 2, 3$, between $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$, correspond to canonical correlation coefficients [25] and are equal to the singular values of the normalized coherence matrix [5]

$$\mathbf{M}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathbf{\Gamma}(\boldsymbol{r}_1)^{-1/2}\Omega(\boldsymbol{r}_1, \boldsymbol{r}_2)\mathbf{\Gamma}(\boldsymbol{r}_2)^{-1/2}, \tag{4}$$

which will be considered with the convention $\mu_{E,1}(\boldsymbol{r}_1, \boldsymbol{r}_2) \geq \mu_{E,2}(\boldsymbol{r}_1, \boldsymbol{r}_2) \geq \mu_{E,3}(\boldsymbol{r}_1, \boldsymbol{r}_2)$. In the following, the analysis will be limited to locations for which the polarization matrices are nonsingular.

If $\mathbf{A}(\boldsymbol{r}_i) = \mathbf{J}(\boldsymbol{r}_i)\mathbf{E}(\boldsymbol{r}_i)$, where $\mathbf{J}(\boldsymbol{r}_i)$ are nonsingular deterministic matrices, then $\mu_{A,j}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mu_{E,j}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ where $\mu_{A,j}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ are the intrinsic degrees of coherence between $\mathbf{A}(\boldsymbol{r}_1)$ and $\mathbf{A}(\boldsymbol{r}_2)$ [5]. The intrinsic degrees of coherence or one-to-one transformations of these quantities are the only parameters of the mutual coherence and polarization matrices that possess this property of invariance [5].

Several interesting properties discussed for 2D fields in [21,22] can be easily generalized to 3D fields when the three intrinsic degrees of coherence are equal to 1.

Property A: The three intrinsic degrees of coherence between the fields $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$ with nonsingular polarization matrices are equal to 1 (i.e., $\mu_{E,j}(\boldsymbol{r}_1, \boldsymbol{r}_2) = 1$ for $j = 1, 2, 3$), if and only if there exists a nonsingular 3×3 deterministic matrix $\mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ such that

$$\mathbf{E}(\boldsymbol{r}_1) = \mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_2)\mathbf{E}(\boldsymbol{r}_2) \tag{5}$$

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in the mean-square meaning, i.e., such that $\langle\|\mathbf{E}(\boldsymbol{r}_1) - \mathbf{J}(\boldsymbol{r}_1,\boldsymbol{r}_2)\mathbf{E}(\boldsymbol{r}_2)\|^2\rangle = 0$ where $\|\|$ is the Euclidean norm.

If the field possesses this property in a spatial domain, $\mathcal{D}$, then the property discussed in [22] can be generalized as well.

Property B: The three intrinsic degrees of coherence between the fields $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$ with nonsingular polarization matrices are equal to 1 in a domain, $\mathcal{D}$, if and only if, in this domain:

(1) the field can be written in the mean-square, meaning

$$\mathbf{E}(\boldsymbol{r}) = \epsilon_1\mathbf{\Psi}_1(\boldsymbol{r}) + \epsilon_2\mathbf{\Psi}_2(\boldsymbol{r}) + \epsilon_3\mathbf{\Psi}_3(\boldsymbol{r}) \tag{6}$$

or equivalently:

(2) the mutual coherence matrix can be written

$$\Omega(\boldsymbol{r}_1,\boldsymbol{r}_2) = \sum_{j=1}^{3}\lambda_j\mathbf{\Psi}_j(\boldsymbol{r}_1)\mathbf{\Psi}_j^\dagger(\boldsymbol{r}_2), \tag{7}$$

where $\lambda_j > 0$, $\mathbf{\Psi}_1(\boldsymbol{r})$, $\mathbf{\Psi}_2(\boldsymbol{r})$ and $\mathbf{\Psi}_3(\boldsymbol{r})$ are geometrically independent deterministic vector fields [i.e., one vector $\mathbf{\Psi}_j(\boldsymbol{r})$ is not the linear combination of the two others $\mathbf{\Psi}_n(\boldsymbol{r})$ and $\mathbf{\Psi}_m(\boldsymbol{r})$], and where ϵ_j are statistical uncorrelated complex random variables (i.e., $\langle\epsilon_j^*\epsilon_k\rangle = 0$ if $j \neq k$).

Mean-square coherent light in 3D in a domain, $\mathcal{D}$, corresponds to fields that satisfy property B and that are thus equal to the sum of three independent (statistically and geometrically) totally polarized fields that satisfy the factorization condition at order one introduced in [26] in the context of quantum optics.

The other limit situation for which the three intrinsic degrees of coherence are all equal to 0 is also worth notice. In that case, the mutual coherence matrix is equal to the null matrix, and no correlation can exist between any component of $\mathbf{E}(\boldsymbol{r}_1)$ and any component of $\mathbf{E}(\boldsymbol{r}_2)$.

Let us now consider the general case where the intrinsic degrees of coherence can have any value between 0 and 1. The properties presented below have been shown in 2D, but their generalization to 3D is not always straightforward, in which case the proofs are given in the Appendices.

A first property [5] is that the maximal modulus of the standard degree of coherence that can be obtained in interference experiments with optimized totally polarized component of the fields $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$ is equal to $\mu_{E,1}(\boldsymbol{r}_1,\boldsymbol{r}_2)$. The generalization of this property to 3D leads to Property C.

Property C: The maximal value of the modulus of the degree of coherence between the totally polarized fields $E_u(\boldsymbol{r}_i)$ obtained with

$$E_u(\boldsymbol{r}_i) = \boldsymbol{u}_i(\boldsymbol{r}_i)^\dagger\mathbf{E}(\boldsymbol{r}_i), \tag{8}$$

is equal to $\mu_{E,1}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ when $\boldsymbol{u}_1(\boldsymbol{r}_1)$ and $\boldsymbol{u}_2(\boldsymbol{r}_2)$, which represent the action of two perfect polarizers, are optimized.

It has been shown [23,24] that the intrinsic degrees of coherence are able to describe the irreversible behavior that exists when fields that fluctuate in 2D are multiplied by random Jones matrices. These results can be generalized to fields that fluctuate in 3D.

Property D: If the fields are modified such that

$$\mathbf{A}(\boldsymbol{r}_i) = \mathbf{J}(\boldsymbol{r}_i)\mathbf{E}(\boldsymbol{r}_i), \tag{9}$$

where $\mathbf{J}(\boldsymbol{r}_i)$ is a random matrix, then the largest intrinsic degree of coherence $\mu_{A,1}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ between $\mathbf{A}(\boldsymbol{r}_1)$ and $\mathbf{A}(\boldsymbol{r}_2)$ satisfies

$$\mu_{A,1}(\boldsymbol{r}_1,\boldsymbol{r}_2) \leq \mu_{E,1}(\boldsymbol{r}_1,\boldsymbol{r}_2). \tag{10}$$

The proof of this property is provided in Appendix A. It is interesting to determine if only $\mu_{E,1}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ is able to describe the irreversible behavior that exists when the fields are modified by random linear transformations represented by matrix vector multiplications. The following property is shown in Appendix B.

Property E: Any continuous and differentiable function that can be determined with second-order statistical characteristics and which value cannot increase with random transformations of Eq. (9) is a one-to-one increasing function of the largest intrinsic degree of coherence.

A more precise result than Property D has been obtained in 2D [24] when the linear random transformations are uncorrelated. Property F, shown in Appendix C, generalizes this result for fields in 3D.

Property F: When the fields are modified accordingly to Eq. (9) with random uncorrelated matrices $\mathbf{J}(\boldsymbol{r}_1)$ and $\mathbf{J}(\boldsymbol{r}_2)$ then, for $j = 1, 2, 3$, the intrinsic degrees of coherence $\mu_{A,j}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ between $\mathbf{A}(\boldsymbol{r}_1)$ and $\mathbf{A}(\boldsymbol{r}_2)$ satisfy

$$\mu_{A,j}(\boldsymbol{r}_1,\boldsymbol{r}_2) \leq \mu_{E,j}(\boldsymbol{r}_1,\boldsymbol{r}_2). \tag{11}$$

It is interesting to note that if $\mathbf{A}(\boldsymbol{r}_1) = \mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{A}(\boldsymbol{r}_2) = \mathbf{J}(\boldsymbol{r}_2)\mathbf{E}(\boldsymbol{r}_2)$ then $\mu_{A,j}(\boldsymbol{r}_1,\boldsymbol{r}_2) \leq \mu_{E,j}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ for $j = 1, 2, 3$, since the conditions of application of Property F are fulfilled. The difference between Properties D and F shows that correlation between the linear random transformations can increase some intrinsic degrees of coherence but not the largest.

Let us illustrate these properties when light propagates in a 3D disordered medium (see Fig. 1) and at locations for which the polarization matrix is nonsingular. A source located at $\boldsymbol{r}_0$ emits the field $\mathbf{E}(\boldsymbol{r}_0)$ with a polarization matrix $\mathbf{\Gamma}(\boldsymbol{r}_0)$ that is nonsingular. The field is thus partially polarized in 3D, and we also assume that the time

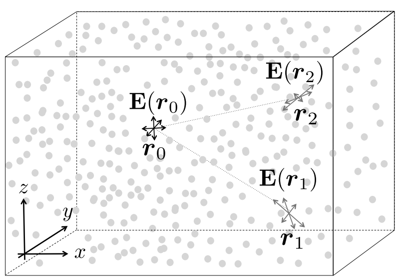

Fig. 1. Illustration of the considered example of light propagation in a disordered medium from a source located at $\boldsymbol{r}_0$ with a nonsingular polarization matrix.

propagation between two locations in the medium is small in comparison to the coherence time of the different components of the vector field $\mathbf{E}(\boldsymbol{r}_0)$, so that the field at location $\boldsymbol{r}_1$ can be written $\mathbf{E}(\boldsymbol{r}_1) = \mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)\mathbf{E}(\boldsymbol{r}_0)$, where $\mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)$ is a matrix. The situations of frozen and nonfrozen disorders will be considered. Frozen disorder corresponds to cases for which the measurements can be performed for one particular realization of the disorder. This is, for example, the case when the disorder does not fluctuate in time, and the coherence is measured between two fixed points. Nonfrozen disorder corresponds to cases for which the measurements are described with averages over the disorder. This is, for example, the case when the disorder fluctuates in time.

Assuming first that the disorder is frozen, the field satisfies Property A in the medium and corresponds to a mean-square coherent field that satisfies Property B. With Eq. (6), the field at location $\boldsymbol{r}$ can thus be written: $\mathbf{E}(\boldsymbol{r}) = \epsilon_1\boldsymbol{\Psi}_1(\boldsymbol{r}) + \epsilon_2\boldsymbol{\Psi}_2(\boldsymbol{r}) + \epsilon_3\boldsymbol{\Psi}_3(\boldsymbol{r})$, where $\boldsymbol{\Psi}_1(\boldsymbol{r})$, $\boldsymbol{\Psi}_2(\boldsymbol{r})$ and $\boldsymbol{\Psi}_3(\boldsymbol{r})$ are geometrically independent deterministic vector fields and where ϵ_j are statistical uncorrelated complex random variables. The intrinsic degrees of coherence between $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_0)$ are all equal to 1, as well as the intrinsic degrees of coherence between $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$ where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are locations in the disordered medium.

On the other hand, when the disorder is not frozen, e.g., as in [16], the relation $\mathbf{E}(\boldsymbol{r}_1) = \mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)\mathbf{E}(\boldsymbol{r}_0)$ is no more deterministic since $\mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)$ now corresponds to a random matrix. In that case, Property F shows that the random linear transformation can result in intrinsic degrees of coherence between $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_0)$ smaller than 1. This is also the case when coherence properties are analyzed at two locations, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, since $\mathbf{E}(\boldsymbol{r}_1) = \mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)\mathbf{E}(\boldsymbol{r}_0)$ and $\mathbf{E}(\boldsymbol{r}_2) = \mathbf{J}(\boldsymbol{r}_2, \boldsymbol{r}_0)\mathbf{E}(\boldsymbol{r}_0)$. Let $\langle\rangle_J$ denote average over the disorder [i.e., over the random matrices $\mathbf{J}(\boldsymbol{r}_i, \boldsymbol{r}_0)$]. With strong fluctuations of the disorder, the average value $\langle\mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)\rangle_J$ can become very small, and the intrinsic degrees of coherence between $\mathbf{E}(\boldsymbol{r}_0)$ and $\mathbf{E}(\boldsymbol{r}_1)$ can also become very small. Furthermore, when the distance between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ increases the matrices, $\mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)$ and $\mathbf{J}(\boldsymbol{r}_2, \boldsymbol{r}_0)$ can become uncorrelated, and the intrinsic degrees of coherence between $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$ can also become very small.

These examples show that different results are obtained if the averages over different disorder realizations of the media (i.e., $\langle\rangle_J$) are applied to the intrinsic degrees of coherence or to the mutual coherence and polarization matrices. Indeed, since the intrinsic degrees of coherence of frozen disordered media are equal to 1, their averages are still equal to 1 while the intrinsic degrees of coherence of the averaged mutual coherence and polarization matrices can be very small. This difference reflects the ability that the fields at two locations can interfere with nonzero visibility when the disorder is frozen, while it is not the case with nonfrozen disorder when the intrinsic degrees of coherence are equal to zero.

Let us finally assume that the field $\mathbf{E}(\boldsymbol{r})$ can be written $\mathbf{E}(\boldsymbol{r}) = \mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}) + \mathbf{E}_{\mathrm{inc}}(\boldsymbol{r})$. The field $\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r})$ is totally polarized and totally coherent and can be generated, for example, with a dipole source located at $\boldsymbol{r}_0$ or, more precisely, $\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}) = \epsilon\boldsymbol{\Psi}_{\mathrm{coh}}(\boldsymbol{r})$, where ϵ is a random variable and $\boldsymbol{\Psi}_{\mathrm{coh}}(\boldsymbol{r})$ is a deterministic vector field. The field $\mathbf{E}_{\mathrm{inc}}(\boldsymbol{r})$ is assumed unpolarized and totally incoherent [i.e., the mutual coherence matrix between $\mathbf{E}_{\mathrm{inc}}(\boldsymbol{r}_1)$ and $\mathbf{E}_{\mathrm{inc}}(\boldsymbol{r}_2)$ is equal to the null matrix when $\boldsymbol{r}_1 \neq \boldsymbol{r}_2$ and the mutual coherence matrix between $\mathbf{E}_{\mathrm{inc}}(\boldsymbol{r}_1)$ and $\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}_2)$ is also equal to the null matrix for any $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$]. In practice, this may be produced by a secondary unpolarized and incoherent source.

The mutual coherence matrix $\Omega(\boldsymbol{r}_0, \boldsymbol{r}_1)$ is then equal to $\langle\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}_0)\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}_1)^\dagger\rangle$ and has thus only one nonzero singular value. However, the polarization matrix at location $\boldsymbol{r}_i$ is $\langle\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}_i)\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}_i)^\dagger\rangle + \langle\mathbf{E}_{\mathrm{inc}}(\boldsymbol{r}_i)\mathbf{E}_{\mathrm{inc}}(\boldsymbol{r}_i)^\dagger\rangle$, which is therefore assumed nonsingular.

Thus, when the disorder is frozen, only one intrinsic degree of coherence between $\mathbf{E}(\boldsymbol{r}_0)$ and $\mathbf{E}(\boldsymbol{r}_1)$ is non-null. This is also the case between $\mathbf{E}(\boldsymbol{r}_1)$ and $\mathbf{E}(\boldsymbol{r}_2)$.

When the disorder is not frozen, the intrinsic degree $\mu_{E,1}(\boldsymbol{r}_1, \boldsymbol{r}_0)$ between the field at locations $\boldsymbol{r}_1$ and $\boldsymbol{r}_0$ can decrease while Property F shows that the two others, $\mu_{E,2}(\boldsymbol{r}_1, \boldsymbol{r}_0)$ and $\mu_{E,3}(\boldsymbol{r}_1, \boldsymbol{r}_0)$, remain equal to 0. The situation is different between two locations, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Indeed, the mutual coherence matrix $\Omega(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is now equal to $\langle\mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)\langle\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}_0)\mathbf{E}_{\mathrm{coh}}(\boldsymbol{r}_0)^\dagger\rangle\mathbf{J}(\boldsymbol{r}_2, \boldsymbol{r}_0)^\dagger\rangle_J$. Since $\mathbf{J}(\boldsymbol{r}_1, \boldsymbol{r}_0)$ and $\mathbf{J}(\boldsymbol{r}_2, \boldsymbol{r}_0)$ can be correlated $\mu_{E,2}(\boldsymbol{r}_1, \boldsymbol{r}_2)$, and $\mu_{E,3}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ can increase and can thus be positive.

To conclude, several coherence properties demonstrated with the intrinsic degrees of coherence for 2D fluctuating fields have been generalized to 3D. These properties have been illustrated with the example of light propagation in 3D disordered media. The concept of mean-square coherent light and the irreversible evolutions of the intrinsic degrees of coherence have been shown to be useful notions for the analysis of coherence properties in disordered media. In particular, clear distinctions between frozen and nonfrozen disorders can be observed. With frozen disorders, applying the average over the disorder to the intrinsic degrees of coherence or to the mutual coherence and polarization matrices can lead to different results with different physical significance. With nonfrozen disorders, the average has to be applied to the mutual coherence and polarization matrices since these average quantities are those that can be measured.

Appendix A: Proof of Property D

In the appendices, the notations are simplified so that $\mathbf{V}(\boldsymbol{r}_1) \to \mathbf{V}_1$ and $\langle\rangle_J = \langle\rangle$. The modulus square of the scalar degree of coherence between $\boldsymbol{u}_1^\dagger\mathbf{A}_1$ and $\boldsymbol{u}_2^\dagger\mathbf{A}_2$ is $\eta_A^2 = |\boldsymbol{u}_1^\dagger\Omega_A\boldsymbol{u}_2|^2/[(\boldsymbol{u}_1^\dagger\boldsymbol{\Gamma}_{A,1}\boldsymbol{u}_1)(\boldsymbol{u}_2^\dagger\boldsymbol{\Gamma}_{A,2}\boldsymbol{u}_2)]$, and Property C lets us know that the maximal value of η_A^2 is $\mu_{A,1}^2$. Equation (9) shows that $\Omega_A = \langle\mathbf{J}_1\Omega_E\mathbf{J}_2^\dagger\rangle$. Equation (4) and the singular value decomposition $\mathbf{M}_E = \mathbf{U}_{E,1}\mathbf{D}_E\mathbf{U}_{E,2}^\dagger$, where $\mathbf{D}_E$ is a diagonal matrix of diagonal values $\mu_{E,j}$ lead to $\boldsymbol{u}_1^\dagger\Omega_A\boldsymbol{u}_2 = \langle\boldsymbol{c}_1^\dagger\mathbf{D}_E\boldsymbol{c}_2\rangle$ with $\boldsymbol{c}_i = \mathbf{U}_{E,i}^\dagger\boldsymbol{\Gamma}_{E,i}^{\frac{1}{2}}\mathbf{J}_i^\dagger\boldsymbol{u}_i$. It can be checked that $\boldsymbol{u}_i^\dagger\boldsymbol{\Gamma}_{A,i}\boldsymbol{u}_i = \langle\boldsymbol{c}_i^\dagger\boldsymbol{c}_i\rangle$ and that $\eta_A^2 = |\langle\boldsymbol{c}_1^\dagger\mathbf{D}_E\boldsymbol{c}_2\rangle|^2/[\langle\boldsymbol{c}_1^\dagger\boldsymbol{c}_1\rangle\langle\boldsymbol{c}_2^\dagger\boldsymbol{c}_2\rangle]$. Furthermore, $\eta_A^2 = \eta_v r_1 r_2$ with $\eta_v = |\langle\mathbf{v}_1^\dagger\mathbf{v}_2\rangle|^2/[\langle\|\mathbf{v}_1\|^2\rangle\langle\|\mathbf{v}_2\|^2\rangle]$ where $\mathbf{v}_i^T = (\sqrt{\mu_{E,1}}c_{i,1}, \sqrt{\mu_{E,2}}c_{i,2}, \sqrt{\mu_{E,3}}c_{i,3})$, and $r_i = \langle\|\mathbf{v}_i\|^2\rangle/\langle\|\mathbf{c}_i\|^2\rangle$. Then $\mu_{A,1}^2 \leq \mu_{E,1}^2$, since $\eta_v \leq 1$ and $r_i \leq \mu_{E,1}$.

Appendix B: Proof of Property E

A quantity F that characterizes the coherence properties at second order has to be a function of $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, and Ω_E, i.e., $F = F(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \Omega_E)$. This quantity can describe an irreversible behavior with random local linear transformations only if it is invariant with deterministic local linear transformations [23], which implies that $F(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \Omega_E) = F(\mathbf{I}_d, \mathbf{I}_d, \mathbf{D}_E)$ where $\mathbf{I}_d$ is the identity matrix in 3D. This relation can also be written [5] $F(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \Omega_E) = f(\mu_1, \mu_2, \mu_3)$ where $\mu_j = \mu_{E,j}$. With random local linear transformations, μ_j are transformed into $\mu_j + d\mu_j$ with $d\mu_1 \leq 0$. If $|d\mu_j|$ are small and $f = f(\mu_1, \mu_2, \mu_3)$, then $df \simeq \sum_{j=1}^{3}(\partial f/\partial \mu_j)d\mu_j$.

However, when $\mu_j < 1$ for $j = 2, 3$, it is always possible to find physical situations and transformations so that $d\mu_j = -\alpha_j d\mu_1$ with α_j arbitrarily large for $j = 2, 3$. This is, for example, observable when Ω_E is diagonal with diagonal elements $I_j\mu_j$ and if the same random rotation of random angle θ is applied to both fields, $\mathbf{E}_1$ and $\mathbf{E}_2$ (see [23], for example, for details in 2D, and the rotation can be applied with axes along the second or third coordinate of $\mathbf{D}_E$). Thus $df \leq 0$ for all physical situations implies that $\partial_{\mu_1} f \geq 0$ and $\partial_{\mu_j} f = 0$ for $j = 2, 3$ and thus that f is independent of μ_2 and μ_3.

Appendix C: Proof of Property F

We consider here random matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ that are uncorrelated. In that case, $\mathbf{M}_A = \langle \mathbf{K}_1 \rangle \mathbf{M}_E \langle \mathbf{K}_2^\dagger \rangle$ with $\mathbf{K}_i = \langle \mathbf{J}_i \mathbf{\Gamma}_i \mathbf{J}_i^\dagger \rangle^{-\frac{1}{2}} \mathbf{J}_i \mathbf{\Gamma}_i^{\frac{1}{2}}$, since $\mathbf{K}_1$ and $\mathbf{K}_2$ are also uncorrelated. Let $\sigma_k(\mathbf{X})$ denote the singular values of the matrix $\mathbf{X}$ ordered by decreasing values. The equality $\sigma_1(\langle \mathbf{K}_i \rangle)^2 = \mathbf{v}^\dagger \langle \mathbf{K}_i \rangle \langle \mathbf{K}_i^\dagger \rangle \mathbf{v}$ holds for some vectors $\mathbf{v}$ with $\|\mathbf{v}\| = 1$. But $\langle \|\mathbf{K}_i^\dagger \mathbf{v} - \langle \mathbf{K}_i^\dagger \rangle \mathbf{v}\|^2 \rangle \geq 0 \Rightarrow \mathbf{v}^\dagger \langle \mathbf{K}_i \rangle \langle \mathbf{K}_i^\dagger \rangle \mathbf{v} \leq \mathbf{v}^\dagger \langle \mathbf{K}_i \mathbf{K}_i^\dagger \rangle \mathbf{v}$ and $\langle \mathbf{K}_i \mathbf{K}_i^\dagger \rangle = \mathbf{I}_d \Rightarrow \|\langle \mathbf{K}_i^\dagger \rangle \mathbf{v}\|^2 \leq 1$. Thus $\sigma_1(\langle \mathbf{K}_i \rangle) \leq 1$.

It can be shown [27] that $\sigma_k(\mathbf{AB}) \leq \sigma_1(\mathbf{A})\sigma_k(\mathbf{B})$. Thus $\sigma_k(\langle \mathbf{K}_1 \rangle \mathbf{M}_E \langle \mathbf{K}_2^\dagger \rangle) \leq \sigma_1(\langle \mathbf{K}_1 \rangle)\sigma_k(\mathbf{M}_E \langle \mathbf{K}_2^\dagger \rangle) \leq \sigma_1(\langle \mathbf{K}_1 \rangle)\sigma_1(\langle \mathbf{K}_2 \rangle) \sigma_k(\mathbf{M}_E)$ or $\mu_{A,k} \leq \mu_{E,k}$ for $k = 1, 2, 3$ since $\sigma_k(\mathbf{M}_E) = \mu_{E,k}$ and $\sigma_k(\mathbf{M}_A) = \mu_{A,k}$.

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